Approximate Methods for Solving Ordinary Differential EquationsAndrea Zhang

Department of Mathematical Sciences, University of Bath

Abstract

This project explores four approximate methods for solving ordinary differential equations (ODEs): Picard's method, Power series, Frobenius method, and asymptotic methods. Through an examples-driven approach, the underlying theory, strengths, and limitations of each technique are investigated. Unusual examples, where standard techniques fail or produce incorrect solutions, provide insights into the limitations of methods and potential research directions. By applying these methods to real-world problems, we derive valuable insights about the natural world. The project also explores the possibility of extending these methods to systems of differential equations and partial differential equations. The comparison and interplay between the methods reveal unique perspectives when solving the same equation, deepening our understanding of approximate methods for ODEs and highlighting possible research directions in this field.

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Chapter 1 Introduction

Differential equations are ubiquitous in the realm of mathematics, serving as the foundation for modelling various real-world phenomena across diverse fields such as physics, biology, economics, and engineering (Zill, 2012). The study of these equations not only aids in understanding the underlying mechanics of these phenomena but also encourages the development of new solutions and techniques that advance scientific knowledge (Braun, 2014). Among the different types of differential equations, ordinary differential equations (ODEs) are the most basic yet essential form, often representing the starting point for more complex investigations. However, finding analytical solutions to ODEs is often challenging or, in some cases, entirely unfeasible (Ablowitz & Fokas, 2003). Consequently, the development of approximate methods for solving ODEs is of critical significance, allowing researchers to gain insights into the properties and behaviour of various mathematical models. This project explores into a comprehensive exploration of some widely used approximate techniques for solving ODEs, namely Picard's method, power series method, Frobenius method, and asymptotic methods. Our investigation follows an example-driven approach that incorporate the introduction of the theory and philosophy underpinning each method, along with their application to standard examples. This approach offers a solid foundation for understanding the strengths and weaknesses of each method, as well as the contexts in which they are most effective (Simmons & Krantz, 2007). Furthermore, we examine unusual examples in which standard techniques may fail or produce incorrect solutions. Investigating these cases offers valuable insights into the limitations and capabilities of each method while bring to light on potential research directions that arise from the study of these unusual examples. To underscore the real-world significance of these methods, we apply them to practical problems found in various scientific disciplines, such as physics and biology. By doing so, we aim to derive interesting facts and insights about the natural world using the techniques developed in this project. As the project progresses, we will explore the possibility of extending these methods to systems of differential equations and approximate methods for solving partial differential equations

(PDEs). Additionally, we will investigate whether employing different techniques on the same equation can yield unique perspectives and insights into the problem. Ultimately, this project not only deepens our understanding of the approximate methods for solving ODEs but also provides valuable knowledge about the limitations, practical applications, and possible research directions in this field.

Chapter 2 Picard's Method

Picard's method, also known as *Picard iteration*, is an iterative method used to approximate solutions to first-order ODEs with an initial condition. The method involves constructing a sequence of approximations that converge to the exact solution by repeatedly solving a sequence of simpler ODEs (Burden & Faires, 2010). The method is based on the idea that if the solution to an ODE is differentiable and the right-hand side of the ODE is locally Lipschitz[1] (Liao, 2012), then the solution can be approximated by a unique sequence of continuously differentiable functions.

Suppose we have on IVP of the form:

$$\frac{dy}{dx} = F(x, y)\ y(0) = y_0$$

The *Picard's method* finds the approximate solutions via the following iterative procedure

$$\begin{aligned} y_0(x) &= y_0 \\ y_n(x) &= y_0 + \int_0^x F(s, y_{n-1}(s))\ \mathrm{ds} \text{ for } n \geq 1 \end{aligned}$$

In this project, we will mention that under suitable conditions, the solution to the initial value problem (IVP) will be $y(x) = \lim_{n \to \infty} y_n(x)$.

[1] A function is said to be Lipschitz if there exists a constant L such that the absolute difference between the function values at any two points is no greater than L times the absolute difference between the points themselves. The Lipschitz constant L is a measure of how fast the function changes.

2.1 Standard Example

The Picard method of successive approximations will first be demonstrated with an IVP whose solution is already known. Consider the IVP with $y(0) = 0$.

$$y' = 2(y + 1).$$

This IVP has the exact solution $y(x) = e^{2x} - 1$.

To solve this IVP using the *Picard's method*, the function $F(x, y) = 2(y + 1)$ and $y_0(x) = 0$. The first approximation $y_1(x)$ can be calculated as follows:

$$y_1(x) = \int_0^x F\big(s_1, y_0(s)\big)\mathrm{ds}$$
$$= \int_0^x 2(y_0(s) + 1)\ \mathrm{ds}$$
$$= \int_0^x 2(0 + 1)\ \mathrm{ds}$$
$$= \int_0^x 2\ \mathrm{ds}$$
$$\Rightarrow y_1(x) = 2x$$

Similarly, $y_2(x)$ can be calculated as follows:

$$y_2(x) = \int_0^x F(s, y_1(s))\mathrm{ds}$$
$$= \int_0^x 2(y_1(s) + 1)\mathrm{ds}$$
$$= \int_0^x 2(2s + 1)\mathrm{ds}$$
$$= 2\int_0^x 2s + 1\mathrm{ds}$$
$$= 2[s^2 + s]_0^x = \frac{(2x)^2}{2!} + \frac{2x}{1!}$$

Therefore, it can be seen by inspection that

$$y_n(x) = \sum_{k=1}^{n} \frac{(2x)^k}{k!}$$

$$\therefore \lim_{n\to\infty} y_n(x) = \sum_{k=1}^{\infty} \frac{(2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} - 1$$

$$= e^{2x} - 1$$

2.2 Limitations and Unusual Examples

The limitations of Picard's method include the fact that it can be time-consuming and computationally expensive for complicated ODEs, and that it may not always converge. Additionally, the method may not be applicable to ODEs with singularities or discontinuities, and it requires the function to be continuous at (0,0). Similarly, if an ODE has a right hand side that is not Lipschitz then the Picard iteration cannot be done

2.2.1 Unusual Example

Consider the IVP where $y(0) = 0$.

$$y' = xy + 2x - x^3$$

Solving this IVP would require the use of a separation of variables and a method of undetermined coefficients, but the Picard's method can also be used.

The functions $y_1(x)$ and $y_2(x)$ can be calculated as follows:

$$y_1(x) = \int_0^x F(s, y_0(s))ds$$
$$= \int_0^x F(s, 0)ds$$
$$= \int_0^x s(0) + 2s - s^3 ds$$
$$= \int_0^x 2s - s^3 ds$$
$$= \left[s^2 - \frac{s^4}{4}\right]_0^x$$
$$\Rightarrow y_1(x) = x^2 - \frac{x^4}{4}$$
$$y_2(x) = \int_0^x F(s, y_1(s))ds$$
$$= \int_0^x s\left(s^2 - \frac{s^4}{4}\right) + 2s - s^2 ds$$
$$= \int_0^x s^3 - \frac{s^3}{4} + 2s - s^3 ds$$
$$= \int_0^x 2s - \frac{s^5}{4} ds$$
$$= \left[s^2 - \frac{s^6}{24}\right] = x^2 - \frac{x^6}{24}$$

It can be seen that:

$$\begin{aligned} y_n(x) &= x^2 - \frac{x^{2n+2}}{4 \cdot 6 \cdots (2n+2)} \\ \lim_{n\to\infty} y_n(x) &= x^2 \end{aligned}$$

This limit holds provided $-1 < x < 1$. This is one of the limitation examples of Picard's method

2.3 Real-World Applications

Despite these limitations, the Picard method is widely used in the numerical solution of ODEs, particularly in cases where exact analytical solutions are not available or are difficult to obtain. It has applications in many fields, including physics, engineering, and economics. For example, in physics, Picard's method can be used to solve differential equations that describe physical phenomena, such as the motion of a particle or the behavior of a fluid.

In engineering, Picard's method can be used to model and control the behavior of dynamical systems such as robots, airplanes, and electrical circuits. For example, Picard's method can be used to approximate the motion of a robotic arm, which can be used in manufacturing processes.

In economics, Picard's method can be used to model the behavior of economic systems that are described by ODEs. For instance, Picard's method can be used to approximate the solution to the differential equations that describe the dynamics of a stock market or the evolution of an economic system over time.

Overall, Picard's method is a powerful mathematical tool that can be applied in various fields to solve complex problems and model dynamic systems.

Chapter 3 Power Series Approximations

The power series method is a versatile mathematical technique used for solving various types of differential equations that cannot always be addressed using conventional methods. This chapter aims to provide an overview of the power series method, its applications, limitations, and the importance of this method in mathematics and other fields.

The power series method assumes that the solution to a given differential equation can be represented as an infinite power series of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where a_n are the coefficients of the power series, x is the independent variable, and x_0 is the center of the series (Olver, 2014), which will be chosen as the location of the initial condition. The method involves substituting the power series into the differential equation and solving for the coefficients a_n. This often leads to a recurrence relation, which can be used to compute the coefficients and construct the power series solution (Zill & Cullen, 2012).

3.1 Standard and Unusual Example

Consider the IVP

$$y' - y = 0 \quad \text{with} \quad y(0) = 1$$

Consider the power series for $y(x)$ as a solution to this IVP

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

The constants $(a_n)_{n\in\mathbb{N}_0}$ are yet to be determined. The complete power series expansions for both x and its derivative can be expressed as:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \text{ and } y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Plugging these expressions into the ODE $\dot{x} + x = 0$ gives

$$[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots] - [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots] = 0$$

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + (4a_4 - a_3)x^3 + \cdots = 0$$

Furthermore, by summation it can be expressed as

$$\sum_{n=1}^{\infty} [na_n - a_{n-1}]x^n = 0.$$

As this expression must hold true for all values of x, the terms inside the square brackets must be equal to 0 for all $n \in \mathbb{N}_0$. This condition leads to the creation of a recursive relationship, which can be expressed as follows:

$$\sum_{n=1}^{\infty} (na_n - a_{n-1})x^n = 0$$

$$\Rightarrow \; na_n - a_{n-1} = 0$$

$$\Rightarrow \; a_n = \frac{a_{n-1}}{n}$$

This implies that we can determine every value of a_n can be found in terms of a_0 as follows:

$$a_n = \frac{1}{n} a_{n-1} = \frac{1}{n(n-1)} a_{n-2} = \frac{1}{n(n-1)(n-2)} a_{n-3} = \cdots \frac{1}{n(n-1)(n-2)(\ldots)(2)(1)} a_0$$

$$\Rightarrow \; a_{\mathrm{n}} = \frac{1}{n!} a_0$$

We can determine the value of a_0 from the initial condition $y(0) = 1$:

$$1 = y(0) = \sum_{n=0}^{\infty} a_n 0^n = a_0 + a_1(0) + a_2(0) + a_3(0) + \cdots = a_0$$

$$\Rightarrow \; a_0 = 1$$

Consequently, we have $a_n = \frac{1}{n!} a_0 = \frac{1}{n!}$ for all $n \in \mathbb{N}_0$. As a result, we can express x as

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Note that this expression for $y(x)$ is exactly e^x, which is indeed the solution to the IVP.

Although this is a valid method for solving ODEs, there are two issues that can arise in many cases:

1. One issue that often arises when using this method is the question of whether the infinite sum of the power series converges, since a power series solution is only valid if it converges.
2. Another potential issue with power series solutions is that they may not have a closed form solution or that the closed form may not be easily recognizable.

3.1.1 Unusual example

Consider the following the IVP

$$y' = x^2 - y^2 \text{ where } y(1) = 1.$$

Let us assume that the function $y = f(x)$ can be represented by Taylor series centred at $x = 1$.

Thus

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

We do not know the function f, so the coefficients $f^{(n)}(1)$ are yet unknown. But the ODE allows us to determine higher derivatives of $y = f(x)$ in terms of previous ones.

We know $y = f(x) = 1$ when $x = 1$

$$f'(1) = y'(1) = 1^2 - 1^2 = 0$$

Derive f gives:

$$f''(x) = y'' = 2x - 2yy'$$

$$f''(1) = 2$$

Similarly,

$$f'''(x) = y'' = 2 - (2yy'' + 2y'y')$$

$$\Rightarrow f'''(1) = -2$$

Lastly,

$$\begin{aligned} f^{(4)} = y^{(4)} = & \quad (2y \cdot y''' + 2y' \cdot y'' + 2y'y'' + 2y''y') \\ = & \quad -6y'y'' - 2yy''' \\ & \quad f^{(4)}(1) = -4 \end{aligned}$$

substituting back gives Taylor series

$$y = 1 + (x-1)^2 - \frac{1}{3}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots$$

This approximates y as a polynomial in x.

The above method carries a very strong assumption is that the solution to the ODE $y = f(x)$ has a well-defined Taylor series (the solution to the ODE is infinitely differentiable) and that the Taylor series converges to $y = f(x)$.

Question: Is there an ODE whose solution is *not* infinitely differentiable? Similarly, an example of an ODE whose solution has a Taylor Series that does not converge to the function? Yes, there are ODEs whose solutions are not infinitely differentiable, and there are also ODEs whose solutions have Taylor series that do not converge to the function itself.

In general, we can do better by replacing "Taylor series" with Power series.

3.1.2 Unusual Example 2

This example is a formal power series with no guaranteed convergence.

$$y' = x^2 - y^2 \quad \text{with} \quad y(1) = 1$$

Assume that y takes the form of a power series in x about $x = 1$, where the constants are yet to be determined

$$y = c_0 + c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^3 + \cdots$$

This example is a formal power series with no convergence.

Then we can use the given information and the ODE to determine an approximate solution

Firstly, replacing $x = 1$ in the approximation for y

$$1 = c_0 + 0 + 0 + \cdots$$
$$\Rightarrow c_0 = 1$$

Then, by differentiating the ODE we obtain

$$y' = c_1 + 2c_2(x-1) + 3c_3(x-1)^2 + 4c_4(x-1)^3 + \cdots$$

From the ODE when $x = 1$

$$y' = 1^2 - 1^2 = 0$$

and by y' we have

$$y' = c_1 + 0 + 0 + \cdots$$
$$y' = c_1$$

combining both results gives:

$$c_1 = 0$$

Upon further differentiation, we get:

$$y'' = 2c_2 + 6c_3(x-1) + 12c_4(x-1)^3 + \cdots$$

$$\text{when } x = 1 \quad y'' = 2c_2$$

Also. From the ODE we have the result below when $x = 1$:

$$y'' = 2x - 2yy' = 2$$

By combining these results, we get:

$$2c_2 = 2$$
$$c_2 = 1$$

Similarly, we have:

$$y''' = 6c_3 + 24c_4(x-1) + \cdots$$
$$y''' = 6c_3$$

Also, from the ODE:

$$y''' = 2 - 2yy'' - 2y'^2$$
$$y''' = -2$$

Therefore, we can obtain the following by combining these two results

$$6c_3 = -2$$
$$c_3 = -\frac{1}{3}$$

Lastly, by the same method:

$$y^{(4)} = 24c_4 + \cdots$$
$$\text{at } x = 1$$
$$y^{(4)} = 24c_4$$

Also, from the ODE:

$$\begin{aligned} y^{(4)} &= -(2y'' + 2yy'') - 4y'y'' \\ &= -6y'y'' - 2yy''' \end{aligned}$$

when $x = 1$:

$$y^{(4)} = -6(0)(2) - 2(1)(2)$$
$$y^{(4)} = -4$$
$$c_4 = -\frac{1}{6}$$

Therefore, an approximate power series is:

$$y \quad = c_0 + (x-1) + (2(x-1)^2 + (3(x-1)^3 + c_1(x-1)^4 + \cdots$$
$$y \quad = 1 + 0(x-1) + \ (\mathrm{x}-1)^2 - \frac{1}{3}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots$$
$$y \quad = 1 + (x-1)^2 - \frac{1}{3}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots$$

3.2 Real-World Applications

The power series method has numerous applications in solving ODEs, linear ODEs, and even some non-linear ODEs (Boyd, 2000). It is particularly useful when other standard methods, such as separation of variables or integrating factors, are not applicable, or when the differential equation does not have a closed-form solution. The power series method has been applied in various fields, including physics, engineering, and economics, to model and analyze complex systems and phenomena (Olver, 2014).

In quantum mechanics, power series are used to solve the Schrödinger equation and other related problems, such as the perturbation theory for finding approximate solutions to problems that cannot be solved exactly (Griffiths, 2004). In signal processing, power series are employed to represent and analyze signals and systems, such as in the Z-transform and the Laplace transform (Oppenheim & Schafer, 2010). In finance, power series are applied to price options and other financial derivatives, such as in the Black-Scholes model and various interest rate models.

These examples demonstrate the versatility and importance of power series in various fields, as they provide a powerful tool for solving problems and analyzing complex systems.

3.3 Limitations

Despite its versatility, the power series method has some limitations. One notable limitation is the convergence of the power series. The power series solution may only converge within a limited radius of convergence around the center x_0, which may not cover the entire interval of interest (Olver, 2014). Additionally, singular points in the differential equation can cause issues in finding a convergent power series solution. In such cases, further techniques, such as the Frobenius method, may be needed to address these issues (Zill & Cullen, 2012). Furthermore, the computation of the power series solution can become complex and cumbersome, especially for higher-order differential equations, making it difficult to obtain explicit solutions in some cases (Boyd, 2000).

Chapter 4 The Frobenius Method

This is a modified power series used for solving ODEs with singular points (i.e. where the coefficients of the ODE are not analytic at that point, when becoming infinite or discontinuous).

The Frobenius method is used to solve linear ODEs of the form:

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are analytic functions (infinitely differentiable in its domain), and the equation has a regular singular point at $x = xo$ (Ince, 1956). The method involves assuming that the solution y can be represented as a Frobenius series of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$$

where a_n are the coefficients of the series, n is a non-negative integer, and r is a non-integer exponent. The coefficients a_n can be determined by substituting the Frobenius series into the ODE and solving for the coefficients. This often leads to a recurrence relation, which can be used to compute the coefficients and construct the Frobenius series solution (Zill & Cullen, 2012).

4.1 Standard and Unusual Example

Consider the Bessel differential equation of order p given by

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \quad \text{for all} \quad x > 0.$$

We apply the Frobenius method, suppose the solution y can be written in the form

$$y = \sum_{m=0}^{\infty} a_m x^{r+m} \text{ with } a_0 \neq 0$$

The aim is to determine suitable expressions for r and a_m.

Substituting this series solution into Bessel's equation and then rearranging gives:

$$a_0(r^2 - p^2)x^r + a_1((r+1)^2 - p^2)x^{r+1} + \sum_{m=2}^{\infty} (a_m(r+m)^2 - p^2) + a_{m-2})\, x^{r+m} = 0$$

Since the right-hand side is zero, all coefficients on left hand side will be zero as well.

Then, we note that, if a_0 is nonzero, then:

$$a_0(r^2 - p^2) = 0$$
$$r = \pm p$$

Similarly, if a_1 is nonzero:

$$a_1((r+1)^2 - p^2) = 0$$
$$r = -1 \pm p$$

This holds more generally for $m \geq 2$:

$$a_m((r+m)^2 - p^2) + a_{m-2} = 0$$

$$\begin{aligned} a_m &= \frac{-a_{m-2}}{(r+m)^2 - p^2} \\ &= \frac{-a_{m-2}}{r^2 + 2rm + m^2 - p^2} \\ &= \frac{-a_{m-2}}{m^2(m+2p)} \end{aligned}$$

Note that the first term implies that $a_0(\mathrm{r}^2 - p^2) = 0$, meaning that either $a_0 = 0$ or $r = -1 \pm p$. On the other hand, the second term $a_1((r+1)^2 - p^2) = 0$ implies that $a_1 = 0$ or $r = -1 \pm p$. This gives rise to two non-trivial cases:

- $a_1 = 0$ and $r = \pm p$
- $a_0 = 0$ and $r = -1 \pm p$

Consider the case when $a_1 = 0$, then we have:

$$0 = a_1 = a_3 = \cdots . = a_{2k+1}$$

For the even coefficients we see that:

$$a_2 = \frac{-a_0}{2(2+p)} = \frac{-a_0}{2^2(1+p)}$$

$$a_4 = \frac{-a_2}{4(4+2p)} = \frac{-a_2}{2^2 \cdot 2(2+p)} = \frac{-a_0}{2^4 \cdot 2(1+p)(2+p)}$$

Therefore,

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!\,(1+p)(2+p) \ldots . (k+p)}$$

Therefore, we obtain the following to Bessel's equation

$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{2^{2k} k!\,(1+p)(2+p) \cdots (k+p)} x^{2k+p}$$

This is called the Bessel function of the first kind of order p.

This solution is denoted by the $J_{\mathrm{p}}(x)$ where n is the order of the Bessel function and x is the argument. The second kind of Bessel functions, also known as Neumann functions, are another set of solutions to Bessel's differential equation. They are denoted by the symbol $Y_{\mathrm{p}}(x)$ and can be derived if it is assumed that $a_0 = 0$, which means that a_1 will be nonzero and following the same process. Therefore, the general solution will be a linear combination of these two solutions, namely:

$$Y(x) = AJ_p(x) + BY_B(x)$$

Where A and B are constants to be found from the initial conditions.

4.2 Real-World Applications

The Frobenius method has numerous applications in solving linear ODEs with regular singular points. These equations frequently appear in various fields, including physics, engineering, and applied mathematics.

For instance, in physics: The Frobenius method is used to solve the radial part of the Schrödinger equation in quantum mechanics, particularly in the study of the hydrogen atom and other atomic systems (Griffiths, 2004). In engineering, specifically in the field of fluid dynamics, the Frobenius method can be applied to solve the Blasius equation, which models the boundary layer flow over a flat plate (White, 2011). Finally, in applied mathematics: The Frobenius method is used in the study of special functions, such as Bessel functions, Legendre functions, and hypergeometric functions, which play crucial roles in the solutions of various mathematical problems and physical phenomena (Olver et al., 2010).

4.3 Limitations

Despite its versatility, the Frobenius method has some limitations. One notable limitation is that the method is only applicable to linear ODEs with regular singular points. It cannot be applied to irregular singular points, and other techniques, such as the Birkhoff-Trjitzinsky method or the WKB method, may be needed to address these cases (Zill & Cullen, 2012). Additionally, the computation of the Frobenius series solution can become complex and cumbersome, especially for higher-order differential equations, making it difficult to obtain explicit solutions in some cases (Ince, 1956).

Chapter 5 Asymptotic Methods

5.1 Introduction

Asymptotic methods are used to approximate solutions of ODEs when an exact solution is difficult to find or not possible. Among these methods, the method of dominant balance plays a crucial role in simplifying and solving ODEs. In this chapter, we will explore the method of dominant balance, its limitations, and its applications with relevant examples.

5.2 Methods of Dominant Balance

The method of dominant balance is a versatile approach for finding approximate solutions to ODEs, particularly when dealing with singular perturbation problems, when a small parameter in a system leads to significant changes in the behavior of the solution (Bender & Orszag, 1999). The technique involves identifying the dominant terms in an equation that balance each other and discarding the non-dominant terms to simplify the problem. This leads to a reduced problem, which can be solved more easily, and the asymptotic solution can be derived from the simplified equation (Bender & Orszag, 1999).

5.2.1 Example

Consider the equation:

$$x^4 y'' = y$$

This has an irregular singular point at $x = 0$.

This is because as x tends to 0, the differential equation reduces to $y = 0$ which eliminates the highest order derivative.

We assume a solution as $x \to 0$ takes the form:

$$y = e^{s(x)}$$

Where *s(x)* is a function to be determined, then by chain rule

$$y'(x) = s'(x)e^{s(x)}$$

Then, by further differentiating it we obtain:

$$y''(x) = (s'^2(x) + s''(x))e^{s(x)}$$

Substituting this into the original equation and then simplifying tells us that

$$x^4(s'^2 + s'') - 1 = 0 \ (**)$$

Assume that the function s takes the form:

$$s'(x) = cx^{\alpha} + A_1(x)$$

Where c and α are constants, and the function A_1 decays to 0 faster than x^{α}. For the purpose of this derivation A_1 will be assumed to be negligible.

Derive the following:

$$s''(x) = \alpha c x^{\alpha-1} + \cdots$$

Substituting this into $(**)$ gives:

$$c^2x^{4+2\alpha} + c\alpha x^{4+\alpha-1} - 1 \sim 0$$

The idea behind method of dominant balance is that two of these terms should be comparable, while the third must be small in comparison to these two. Considering the possibilities:

a) $c^2x^{4+2\alpha} \sim 1$

b) $c^2x^{4+2a} \sim -cax^{4+k-1}$

c) $c\alpha x^{4+\alpha-1} \sim 1$

We can obtain the following expressions:

a) $4 + 2\alpha = 0$

$\Rightarrow \alpha = -2$

b) $4 + 2\alpha = 4 + \alpha - 1$

$\Rightarrow \alpha = -1$

c) $4 + \alpha - 1 = 0$

$\Rightarrow \alpha = -3$

In the last two cases $\alpha = -1$ or $\alpha = -3$ the term that we omit is much larger than the comparable terms as $x \to 0$. So, these choices are inconsistent with the method.

Therefore, we must have $\alpha = -2$ leading to $c^2 = 1$ and thus:

$$c = \pm 1$$

So now we have carried out the first iteration to obtain

$$s'(x) \sim cx^{-2} + A_1(x)$$

substitute this expression into the equation $(**)$. We obtain:

$$x^4(c^2x^{-4} + 2cx^{-2}A_1 + A_1^2) + x^4(-2cx^{-3} + A_1') - 1$$

Simplifying gives:

$$2cx^2A_1 + x^4A_1^2 - 2cx + x^4A_1' \sim 0$$

Again, we let

$$A_1(x) = c_1x^{\beta} + \cdots$$

and repeat a similar argument to find that the only consistent possibility is

$$\beta = -1, \qquad c_1 = 1$$

So now we have:

$$s'(x) = cx^{-2} + x^{-1} + A_2(x)$$

Again, we substitute this to obtain:

$$x^2(1 + 2cA_2) + 2x^3A_2 + x^4A_2^2 - x^2 + x^4A_2' \sim 0$$

As $x \to 0$ this equation is identically satisfied by setting $A_2 = 0$.

So the calculation terminates to give:

$$s'(x) = cx^{-2} + x^{-1}$$
$$s(x) = -cx^{-1} + \log(x) + s_0$$

Therefore, our approximate is:

$$y = e^{s(x)} = e^{-\frac{c}{x}+\log(x)+S_0} = k + e^{\pm\frac{1}{x}}$$

Recalling that:

$$c = \pm 1$$

In this case we obtain an exact solution to the ODE.

5.3 Real-World Applications

The method of dominant balance has been instrumental in a variety of applications across various fields. In fluid mechanics, for instance, it is employed in deriving the boundary layer equations, which describe the flow of viscous fluids over a solid surface. By balancing the dominant terms, the simplified boundary layer equations can be obtained, leading to solutions that capture the essential flow behavior near the surface (Van Dyke, 1975).

In reaction-diffusion systems, the method of dominant balance is used to determine the behavior of the solutions in different regions of the problem domain. For example, the method can help identify the regions where the reaction or diffusion processes dominate the system's behavior (Murray, 2002).

Another notable application of the method of dominant balance is in the study of flame propagation in combustible mixtures. By identifying the dominant balance between the reaction and diffusion terms, researchers can derive simplified equations that describe the behavior of the flame front (Williams, 1985).

5.4 Limitations

The applicability of the method of dominant balance is limited to cases when the ODE exhibits a clear balance between dominant terms, and the neglected terms do not significantly affect the solution (Bender & Orszag, 1999).

In some cases, identifying the dominant terms can be challenging, particularly when the equation involves multiple scales or several competing terms (Nayfeh, 1973).

Furthermore, the dominant terms may change as the independent variable evolves, and multiple dominant balances may occur in different regions of the independent variable (Bender & Orszag, 1999).

Chapter 6 Extending to Systems of Differential Equations and PDEs

In various scientific and engineering fields, systems of differential equations and PDEs frequently emerge, reflecting the complexities of multi-dimensional and multi-variable problems. In this project, we consider extending the approximate methods used for ODEs—Picard's method, power series method, Frobenius method, and asymptotic methods—to systems of differential equations and PDEs.

Systems of differential equations comprise multiple interconnected ODEs and appear in many real-world applications, such as modeling interactions between multiple species in an ecosystem or the dynamics of connected mechanical systems (Boyce & DiPrima, 2012). To extend the approximate methods studied for ODEs to systems of differential equations, these techniques must be adapted to handle the complexities of multiple, interdependent equations. For instance, Picard's method could be adjusted to use successive approximations for each equation in the system, while the power series and Frobenius methods may need multivariate expansions (Pozrikidis, 2011).

On the other hand, PDEs involve multiple independent variables and derivatives of varying orders. They are commonly used to describe physical phenomena, such as heat conduction, wave propagation, and fluid flow (Strauss, 2008). Extending the approximate methods for ODEs to PDEs presents a more difficult challenge due to increased mathematical complexity. However, some techniques, like asymptotic methods, have already been widely applied to PDEs, especially for boundary value problems and perturbation theory (Holmes, 1995). Other methods, such as the power series and Frobenius methods, may need significant modifications or adaptations for PDEs.

Chapter 7 Comparative Analysis of Different Methods

In this project, we have examined four approximate methods for solving ODEs—Picard's method, power series method, Frobenius method, and asymptotic methods—with each possessing its own advantages, disadvantages, and areas of application. Conducting a comparative analysis of these methods is essential for understanding their relative performance and suitability for various problems.

Picard's method, an iterative technique based on successive approximations, is especially helpful for determining the existence and uniqueness of solutions for specific ODE types (Pozrikidis, 2011). However, its convergence can be slow, and it may not be suitable for nonlinear problems or systems with strong oscillations.

The power series method uses Taylor series expansions to approximate ODE solutions near a given point (Boyce & DiPrima, 2012). This method is versatile and applicable to a wide range of problems, but its accuracy and convergence might be limited by singularities or the choice of expansion point.

The Frobenius method enhances the power series method by allowing solutions with non-integer exponents (Coddington & Levinson, 1955). It is particularly effective for problems with singular points but may be less suitable for problems without such points or with irregular singularities.

Asymptotic methods, such as the matched asymptotic expansions method, are valuable for problems with small or large parameters and boundary layers (Holmes, 1995). These methods provide accurate approximations in many situations where other methods may struggle, but their applicability can be limited by the presence of multiple scales.

In conclusion, each method offers distinct advantages and limitations, making them suitable for specific problems. A thorough understanding of these methods and their comparative performance is essential for selecting the most appropriate technique for a given problem and developing novel methods that address existing limitations

Conclusion

In summary, this project has thoroughly examined four key approximate methods for solving ordinary differential equations: Picard's method, power series method, Frobenius method, and asymptotic methods. By investigating the fundamental theory, principles, standard examples, unusual examples, and real-world applications of each method, we have gained a deep understanding of their strengths, limitations, and the situations in which they are most effective (Simmons & Krantz, 2007). The example-driven approach used in this project has helped us identify the advantages and disadvantages of each method, setting the stage for future research and practical applications in the field of differential equations.

Furthermore, our exploration of unusual examples and the limitations of each method has opened potential research opportunities, promoting the development of new techniques and enhancements to existing methods (Arnold, 2006). Applying these techniques to real-world problems in various scientific disciplines has emphasized their practical importance and showcased their ability to offer valuable insights into the workings of the natural world.

Through the project, we evaluated the possibility of extending these methods to systems of differential equations and partial differential equations, thus broadening their applicability (Ablowitz & Fokas, 2003). We also investigated the potential insights that could be gained from using different techniques on the same problem, revealing unique perspectives, and improving our understanding of the issue at hand.

In conclusion, this project has not only deepened our understanding of approximate methods for solving ODEs but also provided essential knowledge about their limitations, practical applications, and potential research avenues in this field. The experience and understanding gained through this investigation will undoubtedly be valuable for future endeavors.

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