A Brief Overview of Special Functions and Their Applications

Year 3 BSc Project

23038

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Abstract

There are many ‘special functions’ including those from intuitive and straightforward examples to more complicated functions that are intertwined with some of the most confounding and still developing areas of mathematics. These functions have been used in relation to applied problems such as ordinary differential equations, and can be utilised further to make sense of abstract concepts. Additionally, they have the potential to be helpful and solve computational barriers within statistics, quantum mechanics, string theory, and engineering. As a result, this can lead to many real life applications which will be explored throughout this project. The aim is to give an overview of how versatile these special functions are and demonstrate applications in a wide range of scenarios, including seemingly unrelated topics such as rainbows and communication between submarines.

Contents

Chapter 1: Introduction

Special functions are functions that are not defined in the traditional sense and therefore fall under many different categories and can appear in many forms. Some are simple and piecewise defined, others are defined in terms of prime numbers, as analytic extensions of discrete functions, or as solutions of ODEs.

In Chapter 2 four functions with fairly simple definitions will be investigated, namely the Sign function (or alternatively, Signum function to avoid confusion with the trigonometric Sine), the Heaviside step, which follows onto the ramp function which is of a similar nature, and the Dirac delta function. All of these functions have real life applications, and the absence of complicated definitions means it is possible to express these functions in terms of each other, which will be shown explicitly.

Chapter 3 explores the Gamma function which was created as an analytic extension of the commonly used factorial function. The motivation behind this development was to expand the domain from only non-negative integers to the complex numbers. The Gamma function can also be related to the Beta function. Both of these functions have distributions defined in statistics, and these can also be shown to have connections to other heavily used distributions. The Gamma function can also be related to the zeta function, which plays a crucial part in the famous and still unproved Riemann Hypothesis, which will be explored briefly.

The Airy and Bessel functions, which are the focus of Chapter 4, are the solutions to their related ordinary differential equations and have many applications, especially in physics. This includes the use of Airy functions in distinguishing supernumerary fringes in rainbows, which was their original purpose, and the advantages of Bessel beams as a result of their unusual property. This advantageous nature of Bessel beams then leads to modern applications in developing medical equipment used in hospitals and improving signals under the sea.

Finally, in Chapter 5, we will look at functions which are more related to pure mathematics, namely number theory, and heavily involve prime numbers. This will then lead to a demonstration of graph theory from the Möbius function by exploring different definitions of edge sets and the graphs that arise as a result. An argument for why conjectures or hypotheses may not be true even if they seem to hold for all tested numbers is also shown following from the involvement of the Liouville function.

Chapter 2: Generalised Distribution Functions

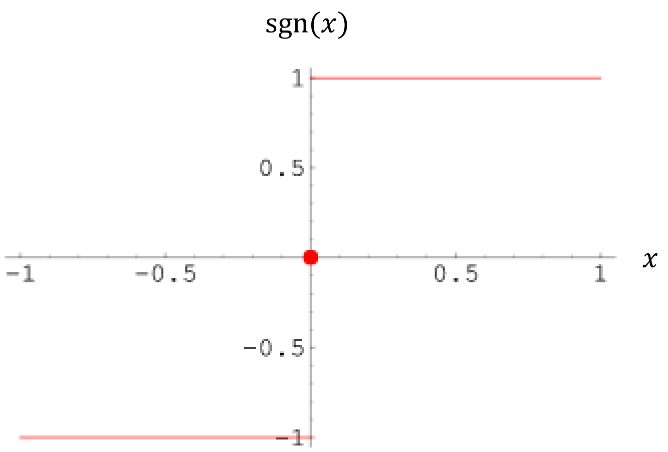
In this Chapter, the Sign, Heaviside step, ramp, and Dirac delta functions will be introduced with definitions and graphs as well as some examples of their common applications. This will then allow us to link them together with equations.

2.1 Sign Function

The *sign*, or *signum*, function is denoted as sgn and defined as follows, for $x \in \mathbb{R}$:

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}$$

This means simply that the sign of a real number is 1 for a positive number, -1 for a negative number, and 0 for the integer zero, and can be plotted as seen below:



(Weisstein, 2022)

This function provides a quick way of defining the sign of a real number and can also be applied to vectors and matrices componentwise. The sign function can extended to the complex numbers (Porubsky, 2006), so, for $z \in \mathbb{C}$:

$$\operatorname{sgn}(z) = \begin{cases} \dfrac{z}{|z|} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$

As a result, this means that the sign function projects a complex number (provided $z \neq 0$) onto the unit circle.

The sign function has many practical applications (Ranguwar, 2021), for example in probability theory where it can be used to determine if an event happens or not, defining 1 as the event occurring, and -1 as not occuring, and therefore is applicable to any scenario with two mutually exclusive outcomes including, for example, situations modelled as having an outcome of either success or failure. Additionally, it is also used in thermodynamic systems which include simple thermostats, where the sign function can be transformed to a certain value, and when this value is reached the system can be activated and begin heating or cooling a room/building as necessary. The theory can be applied to any similar situation where there is a set deterministic value, where the input can be continuously monitored by the function.

2.2 Heaviside Function

The *Heaviside step* function, denoted H, is formally defined as follows, for $x \in \mathbb{R}$:

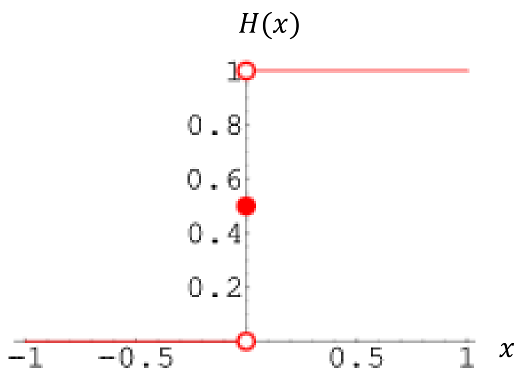
$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.5 & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}$$

It can be linked to the sign function by the following relation, for all $x \in \mathbb{R}$:

$$\text{sgn}(x) = 2H(x) - 1.$$

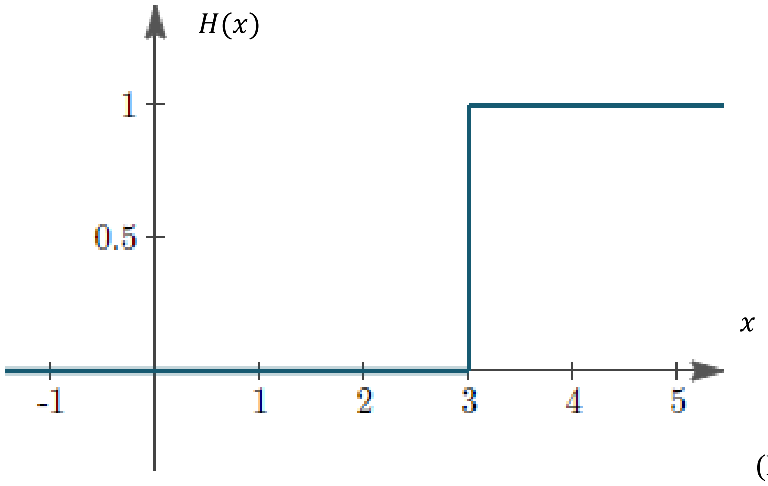
(Weisstein, 2022)

In this case, the value of 0.5 at $x = 0$ has been introduced to address the discontinuity, as the Fourier series expansion of the periodic extension of the Heaviside function without a defined value for $x = 0$ would converge to the midpoint, namely 0.5. This is not always present in other sources and definitions, but the structure of the function remains the same in either case, and will be the definition used for this project. The Heaviside function can be plotted as follows:



(Weisstein, 2022)

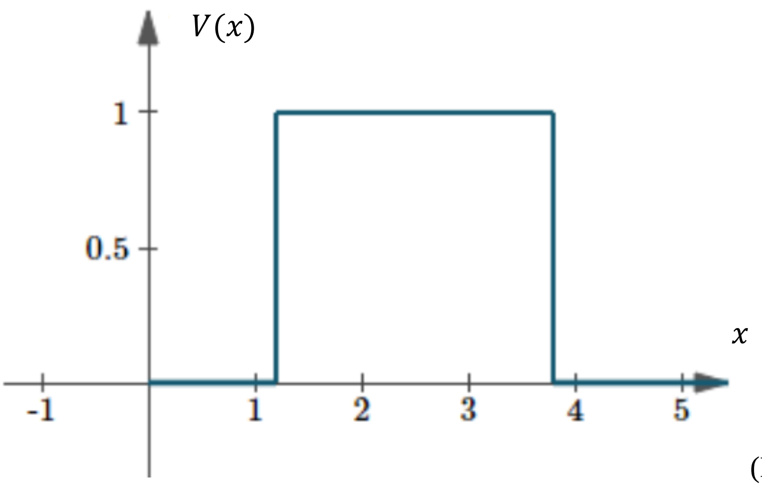
The Heaviside step function can also be shifted so the discontinuity is placed at a different value (Bourne, 2021). The advantage of this is that when the function is being applied to usage in electrical engineering, a circuit can be sketched using the function to graph a delayed waveform. An example is shown below, where the Heaviside step function is equivalently named as a shifted unit step function. Here, the discontinuity is located at the value of $x = 3$:



(Bourne, 2021)

The above could equivalently be denoted using the notation $H_3(x) = H(x-3)$ for all $x \in \mathbb{R}$.

The Heaviside function can also be used to represent a rectangular pulse, where the waveform is later removed after being introduced and thus the function goes back to its original value of zero, and therefore represents a scenario such as a voltage or current passing through a circuit for a certain amount of time. An example with the time parameter of seconds and a duration of 2.6 (from 1.2 to 3.8 seconds) can be given by $V(x) = H(x-1.2) - H(x-3.8)$ as shown below:



(Bourne, 2021)

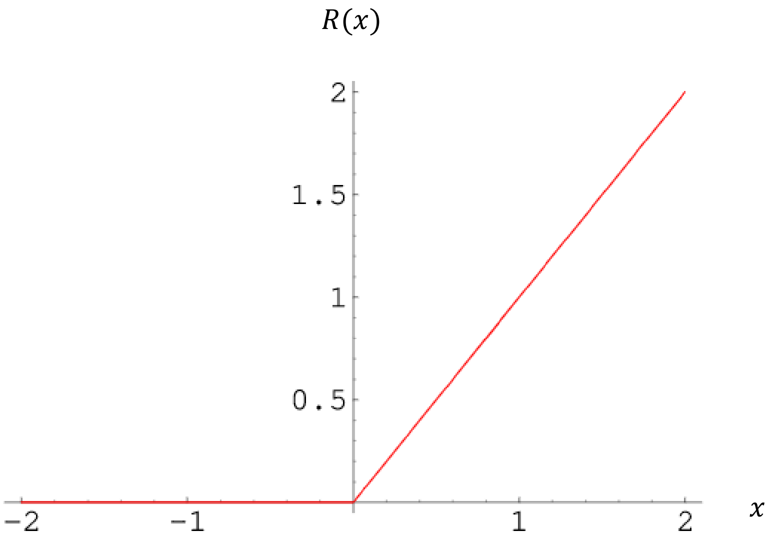
The Heaviside function has other real life applications such as in control theory (Baowan, 2017, pp.35-58), where it represents when a signal turns on and then stays on, and therefore graphs that are similar to those seen above are used for signal processing. This again would require a time scale, most commonly with the parameter of seconds on the horizontal axis, so the time when the signal comes on can be easily read off. More specifically, in broadcast engineering, when hours of footage is being transmitted, the step up helps immediately spot where errors occur and therefore with an actual time stamp recovered, this can be translated across other equipment to locate where the error happened, help to resolve it, and investigate how to prevent the same problem happening again in the future.

2.3 Ramp Function

Another function that could perhaps be argued to follow from the nature of the Heaviside step function is the *ramp* function, defined for $x \in \mathbb{R}$:

$$R(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x & \text{for } x \geq 0. \end{cases}$$

This can be graphed as follows:



(Weisstein, 2022)

In fact, the ramp function can be represented in terms of the Heaviside function in several ways for $x \in \mathbb{R}$, as shown:

- $R(x) = x\, H(x)$
- $R(x) = \int_{-\infty}^{x} H(x')\, dx'$
- $R(x) = \int_{-\infty}^{\infty} H(x')H(x - x')\, dx'$
- $R(x) = (H * H)(x)$

Therefore, according to Weisstein, the ramp function can be expressed in terms of a convolution of $H(x)$ ($*$ denotes convolution in the final line).

The shape of the ramp function is advantageous as it allows the inclusion of systems which have a signal that is gradually applied, rather than instantaneously, and consequently widens the scenarios that can be illustrated. In electrical circuits it also allows the possibility of a variable resistor to increase (or decrease) uniformly over a set amount of time.

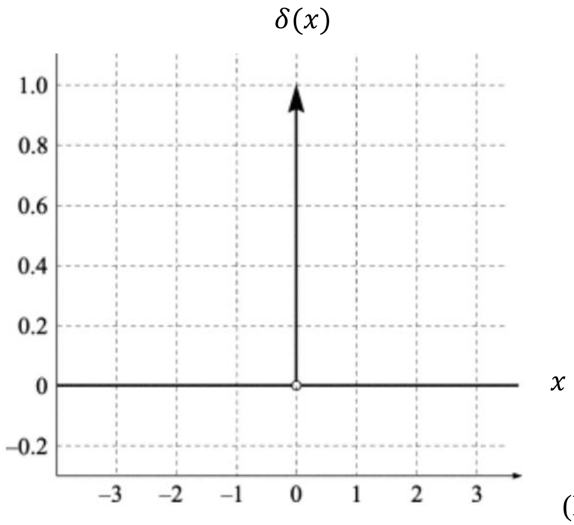
2.4 Dirac Delta Function

In addition to being able to be expressed in terms of the sign and ramp function, the Heaviside function has the weak derivative (the discontinuity at $x = 0$ means it is not a derivative in the traditional sense) of the *Dirac delta* function, for $x \in \mathbb{R}$ (Champeney, 1989, p.127):

$$\frac{d}{dx}H(x) = \delta(x).$$

The Dirac delta function is defined for $x \in \mathbb{R}$ and graphed as:

$$\begin{cases} \delta(x) \to \infty & \text{for } x = 0 \\ \delta(x) = 0 & \text{for } x \neq 0. \end{cases}$$

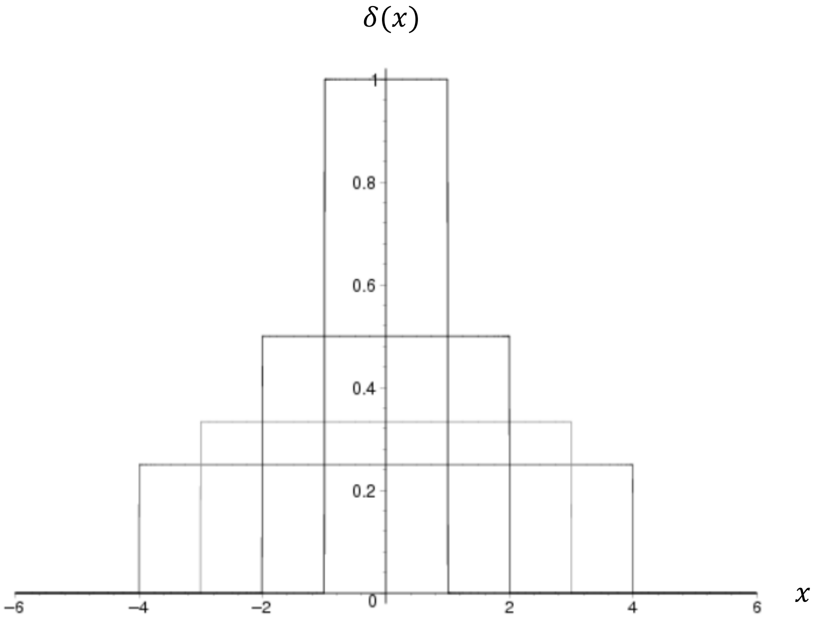


(Baowan, 2017, pp.35-58)

The Dirac delta function has a number of interesting and consequently useful properties. Somewhat surprisingly, the area under the function is always 1 (Dray, Manogue, 2017):

$$\int_a^b \delta(x)\, dx = 1 \text{ for } a < 0 < b.$$

This can be seen by taking a functional sequence that tends to the Dirac delta function, starting from rectangles centred about $x = 0$:



(Dray, Manogue, 2017)

A sketch of an informal proof of this scenario would be to consider a sequence of rectangles with width ε and height $\frac{1}{\varepsilon}$ where $\varepsilon \neq 0$, and therefore the area of each rectangle is 1. As $\varepsilon \to 0$, the sequence of rectangular functions will tend to the Dirac delta function and the area, which is independent of ε, will remain unchanged. This limiting process is not uniformly convergent but demonstrates the reasoning behind the argument.

Similar to the Heaviside step function, the discontinuity can be shifted to any value $a \in \mathbb{R}$, to obtain $\delta_a(x) = \delta(x - a)$. The Dirac delta is heavily used in statistics (Khuri, 2004, pp.1-12), and can be included in alternative representations of fairly simple concepts such as transformations of discrete random variables and binomial expansions. This is achieved by using the transformational property of the function to shift moments of variables to non-central positions, and removes the need to define additional variables to address these movements, and in fact reduces the process to be more direct, as proven by Khuri.

Furthermore, the Dirac delta function is used in physics, more specifically quantum mechanics (Zimmerman Jones, 2019). The nature of the function means it can be used to represent the idea of a point mass in mechanics, or a point charge in electrical systems, as the function provides infinite mass/charge at the specified point and a value of zero for all other real numbers. In addition, it can be shifted to represent an instantaneous impulse at a certain point in time. The reason for using the Dirac delta function in these situations is that it makes integrals over regions that have zero defined almost everywhere make sense, due to the property of the area being 1. This enables calculations containing integrals and can even be extended to more complicated examples that involve higher dimensions, and therefore can be applied to the three dimensional scenario, rather than strictly theoretical and simplified cases.

2.5 Relationships Between Functions

The sign, Heaviside step, ramp and Dirac delta functions are all fairly simple to define and can be linked together by reasonably simple equations involving just basic mathematical operations and calculus. However, we have seen that the simplicity of these functions allows them to make sense of much more complicated concepts and areas of mathematics, as well as redefining some topics in new notation.

The table below shows the relationships between the four functions, for $x \in \mathbb{R}\backslash\{0\}$, by using combinations of the already stated equations and using weak derivatives to address discontinuity at $x = 0$:

	Sign	Heaviside step	Dirac delta	Ramp
Sign	$\operatorname{sgn}(x)$	$2H(x) - 1$	$2\int_{-\infty}^{x} \delta(x)dx - 1$	$\frac{2R(x) - 1}{x}$
Heaviside step	$\frac{\operatorname{sgn}(x) + 1}{2}$	$H(x)$	$\int_{-\infty}^{x} \delta(x)dx$	$\frac{R(x)}{x}$
Dirac delta	$\frac{d}{dx}\frac{\operatorname{sgn}(x) + 1}{2}$	$\frac{d}{dx} H(x)$	$\delta(x)$	$\frac{d}{dx}\frac{R(x)}{x}$
Ramp	$\frac{x \operatorname{sgn}(x) + 1}{2}$	$x\, H(x)$	$x \int_{-\infty}^{x} \delta(x)dx$	$R(x)$

The table below clarifies, for reference, the locations of the discontinuities in each function as these are defining features:

	Sign	Heaviside step	Dirac delta	Ramp
Location of discontinuity for graph without shifts	$x = 0$	$x = 0$	$x = 0$	No discontinuity

Chapter 3: The Gamma Function

This Chapter will focus on the Gamma function, starting with its direct applications, and then will go further to look at the applications of different functions that are closely related, namely the Riemann zeta and Beta functions.

3.1 Introduction

An example of a function more complicated to define is the *Gamma* function, which is based on the concept of an analytic continuation to the factorial function, from the non-negative integers, to complex numbers, and is the work of Leonhard Euler (Müller, 2021).

For reference, the factorial function, for all $n \in \mathbb{N}$, is:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1.$$

Euler first defined the factorial function in terms of the integral (verification in Appendix A) for all $n \in \mathbb{N}$:

$$n! = \int_0^1 (-\ln(x))^n \, dx,$$

By utilising the substitution:

$$x = \mathrm{e}^{-t}$$

The integral becomes, for all $n \in \mathbb{N}$:

$$n! = \int_0^\infty t^n \, \mathrm{e}^{-t} \, dt.$$

(Müller, 2021)

Let the integral above be defined as the function $\Pi(n)$ for $n \in \mathbb{N}$.

It can be proven through using integration by parts and induction (shown in Appendix B) that this function, $\Pi(n)$, is indeed equivalent to the original factorial defined above for positive integers.

This integral can be extended to include the complex numbers and therefore define the Gamma function, which is seen below, for $z \in \mathbb{C}$:

$$\Gamma(z) = \int_0^\infty t^{z-1} \, \mathrm{e}^{-t} \, dt.$$

(Müller, 2021)

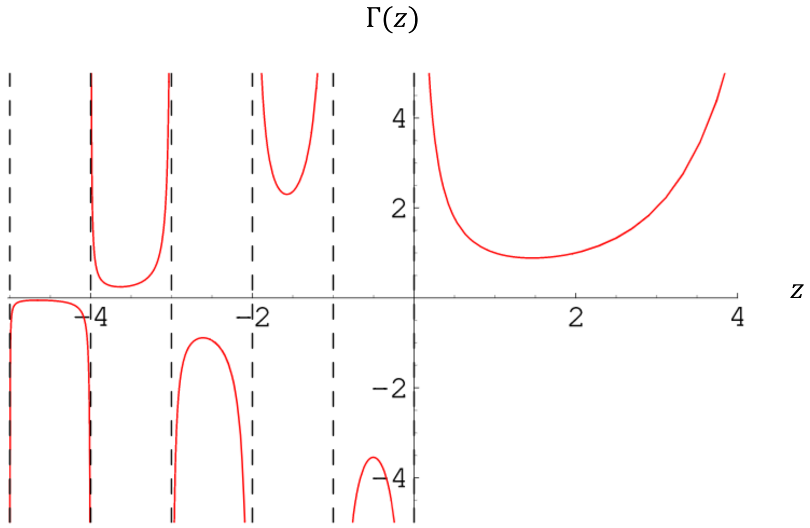
From the expressions of Γ and Π, it can be seen that for any $n \in \mathbb{N}$:

$$\Gamma(n) = \Pi(n-1) = (n-1)!.$$

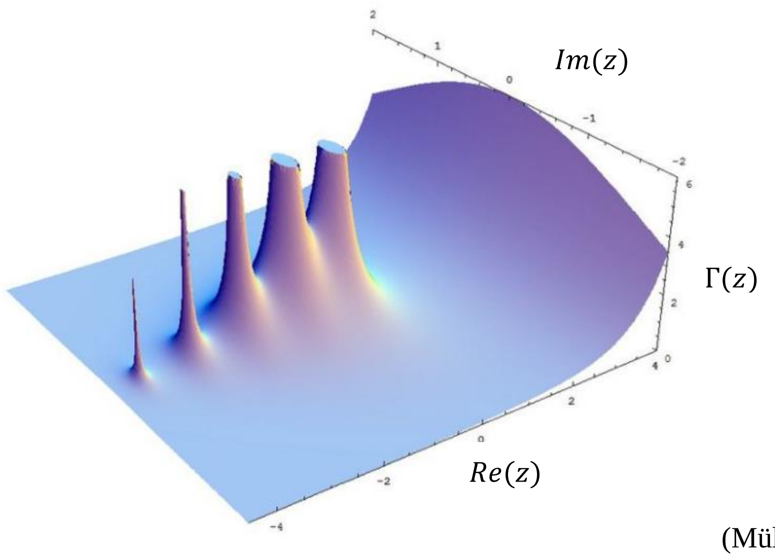
(Müller, 2021)

The Gamma function is analytic (complex differentiable) at all points in $\mathbb{R}$, with the exception being the set encompassing zero and the negative integers, and there are no points where $\Gamma(z) = 0$ (Weisstein, 2022).

The function is graphed, in the real and complex domains as follows:



(Weisstein, 2022)



(Müller, 2021)

3.2 Application of the Gamma Function in Statistics

The Gamma function can model continuous change due to its domain being extended to a continuous set (Gregersen, 2017).

In statistics, the Gamma distribution, which includes the Gamma function in its probability density function (PDF) for positive values of x, is commonly used (Hossein, 2014). The Gamma distribution has two parameters, the shape parameter $\alpha > 0$, and the rate parameter $\lambda > 0$. The distribution is represented by:

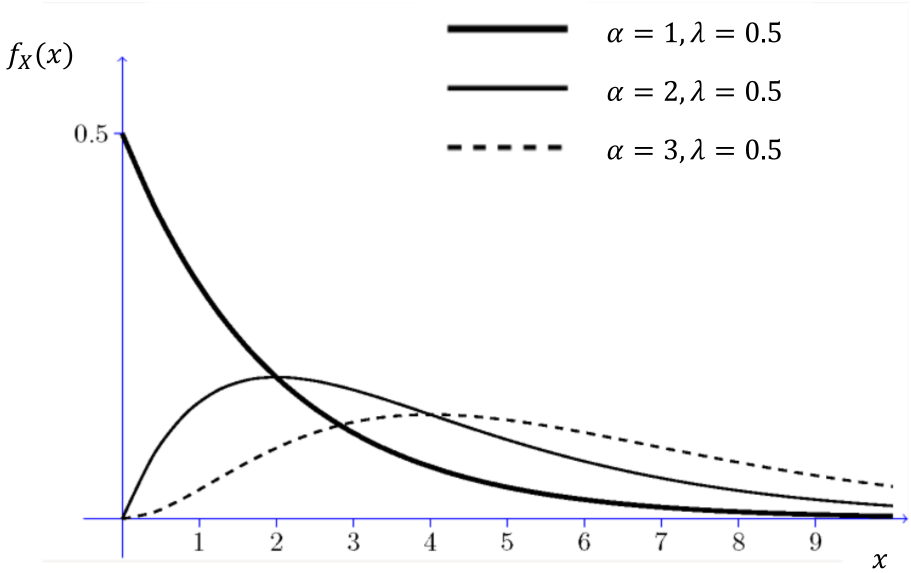
$$X \sim \text{Gamma}(\alpha, \lambda).$$

Provided $x > 0$, the PDF is given by:

$$\frac{\lambda^{\alpha} x^{\alpha-1} \mathrm{e}^{-\lambda x}}{\Gamma(\alpha)}.$$

(Hossein, 2014)

Some examples of Gamma distributions for different values of α and a constant λ are plotted here:



(Hossein, 2014)

By replacing $\alpha = 1$ into the PDF, it can be seen that a Gamma(1,λ) distribution is equivalent to the exponential distribution, Exp(λ). It also follows that a Gamma(*n*,λ) distribution is equivalent to the sum of *n* different Exp(λ) distributions, under the assumption that the observations are independent.

As well as the exponential distribution, there is also a relationship between the Gamma and the normal distribution (Taboga, 2017), which can be seen by transforming a centred normalised N(0,1) normal distribution as follows:

1. Collate *n* normally distributed random variables, which must have zero mean and be mutually independent
2. Square each individual normal variable
3. Sum the squares over *n*.

The Gamma distribution can also be applied to find the moment generating function of the normal distribution provided it has a zero mean, and the extensive derivation is shown in the paper (Iddrisu, Tetteh, 2017, p.7). The moment generating function is often useful to calculate the variance as it can be expressed as $E(X^2) - E(X)^2$, and in many scenarios this reduces computation time compared to other definitions.

Another application of the Gamma distribution is in modelling stochastic hyperelastic materials under radially symmetric and uniform deformations (Alamoudi, Mihai, 2021) or large tensile loading (Fitt, Mihai, Woolley, 2019). The elasticity of a hyperelastic tube or spherical shell are assumed to follow a Gamma distribution and the effect that has on the shearing and inflation is studied to characterise the amplitude and period of any oscillations, alongside any increases or decreases in radius. In this area of research there are also demonstrations of a Gamma distribution converging to a normal distribution, as discussed above (Alamoudi, Mihai, 2021, p.12). The Gamma distribution is appropriate for this application as the elastic properties need to be probabilistic and positive.

3.3 Riemann Zeta Function

The *Riemann zeta* function can be defined as follows for $s \in \mathbb{C}$:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

(Duncan, 2013)

This formula converges for values where $\text{Re}(s) > 1$ and therefore this can be taken as the domain. The zeta function has a closed form for all even positive integers, whereas for odd integers the result is still finite but there is no well-known closed form.

The Riemann zeta function and the Gamma function can be linked by the relation:

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \frac{x^{s-1}}{\mathrm{e}^x - 1}\,dx.$$

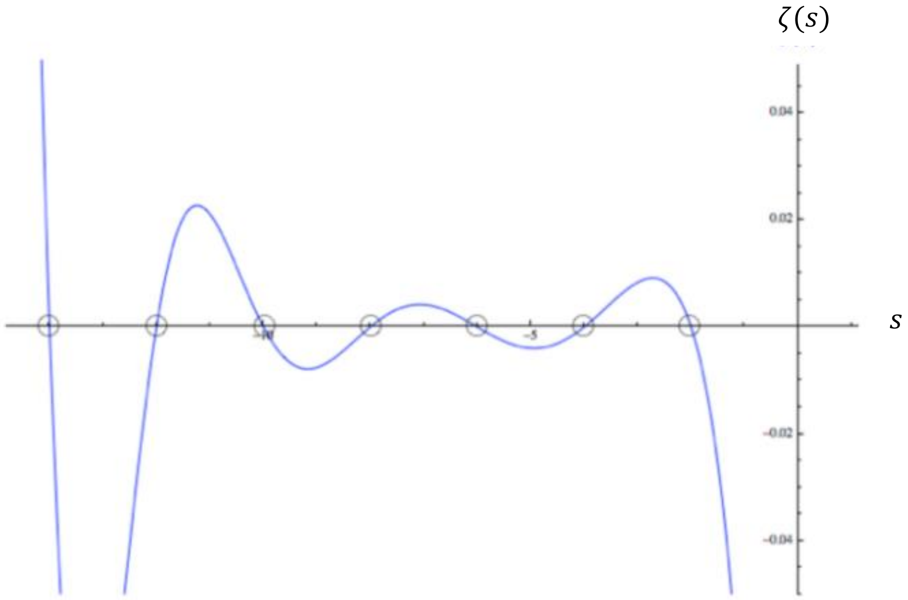
(Müller, 2021)

This equation is again valid for values of $s > 1$ as a result of the convergence. Another relation is the more complicated equation below, also for $s > 1$, with the derivation shown by Müller:

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

From this equation the existence of zeros of the zeta function can be seen, as $\Gamma\left(\frac{s}{2}\right)$ has singularities at the values of any even, negative integers, and consequently will blow up, and therefore to keep the balance between the left and right hand side of the equation the zeta function must take a value of zero at these points to counteract the dramatic increase from the singularities.

The Riemann zeta function has the important application of playing a fundamental role in the famous Riemann Hypothesis which still hasn't been proven to this day, and has a reward of one million US Dollars for whoever proves it. The Riemann Hypothesis was first opened to potential proofs in 2000 as one of the millennium prize problems by the Clay Mathematical Institute (Sarnak, 2004). The plot of the Riemann zeta function for $z < 0$ is shown below:

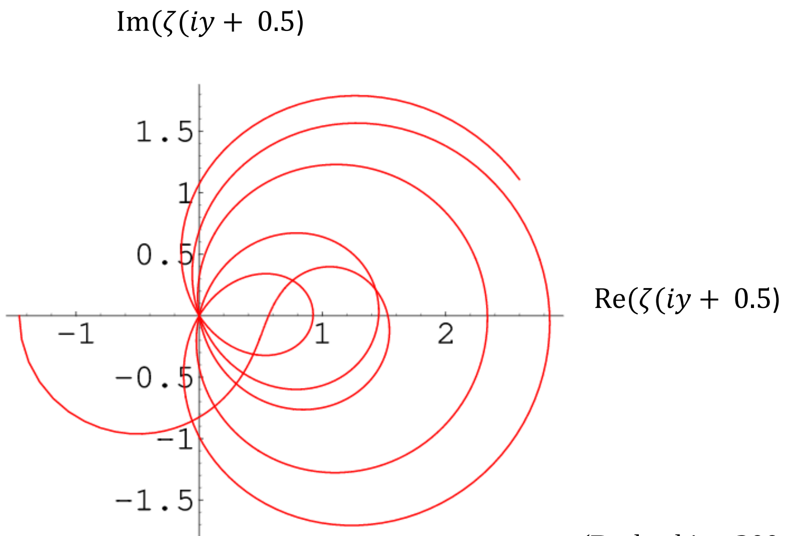


(Veisdal, 2016)

The zeroes are located at the even, negative integers, as circled above.

The Riemann Hypothesis states that any non-trivial zeros (not at an even, negative integer) of the Riemann zeta function will always have real part equal to a half, i.e. if $\zeta(s) = 0$, then $\mathrm{Re}(s) = 0.5$.

Consequently, this line is given the name "the critical line" (Veisdel, 2016) and is known to hold for the first 250 billion roots (Weisstein, 2022). The graph below illustrates values of $\zeta(s)$ along the critical line for $0 \leq y \leq 35$.



(Derbyshire, 2004, p.221)

3.4 Beta Function

Alongside the Gamma function, Leonhard Euler also discovered the related *Beta* function (Weisstein, 2022), which is defined as:

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1 - t)^{\beta-1}\, dt,$$

where $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$. By a change of variables it can be seen that the Beta function is symmetric:

$$B(\alpha, \beta) = B(\beta, \alpha).$$

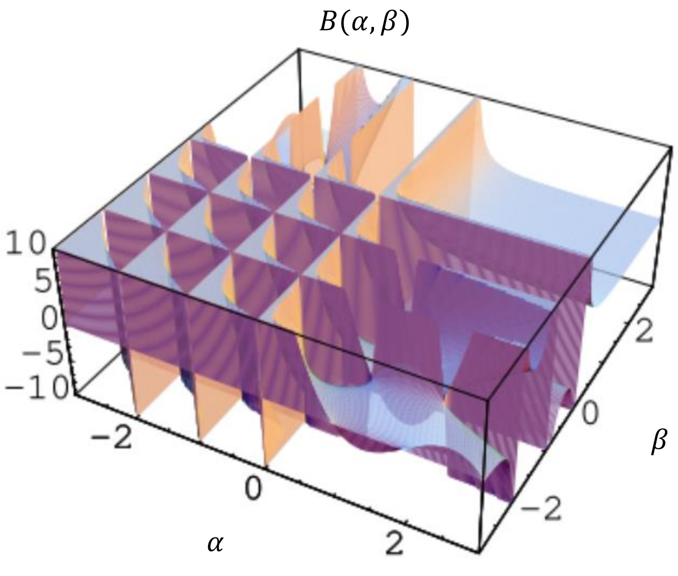
(Choi, Srivastava, 2012)

The Beta function can be written in terms of the Gamma function as follows:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

(Artin, 1931, pp.18-19)

The Beta function can be represented graphically in three dimensions:



(Weisstein, 2022)

The Beta function was used in 1968 by Gabriele Veneziano, a theoretical physicist who was investigating interactions between mesons, which are subatomic particles consisting of quarks and antiquarks (Müller, 2021). Veneziano, who was conducting research projects at CERN, found that the Beta function that had originally only been used in pure mathematics since its conception seemed to demonstrate many properties of the strong nuclear force, which affects subatomic particles including mesons (Riddhi, 2010).

After this discovery, many more institutions began using the Beta function as a representation of a one dimensional string and therefore could be considered the first use, or in other words, the origin of string theory, which has since developed to become an extremely prevalent area of interest in particle physics in the modern day.

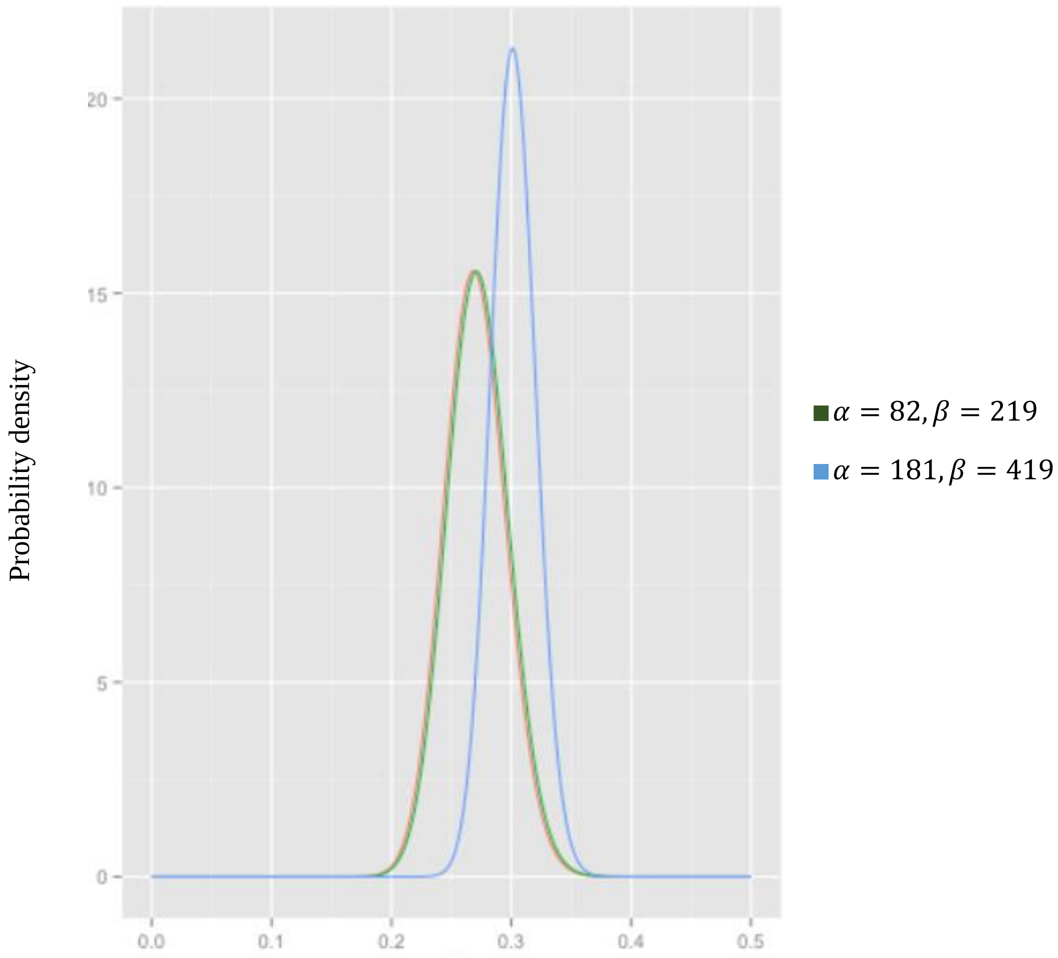
In probability, the Beta distribution can be used to model the likelihood of a certain event occurring, and as such is bounded and has a domain of [0,1] (Kim, 2020).

For both α and β defined as positive real values, the PDF of the Beta distribution is given by:

$$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}.$$

(Kim, 2020)

An example of the PDF for some values of α and β is shown below:



x

(Robinson, 2014)

The Beta distribution shows some resemblance to the binomial distribution, with the difference between them being that whilst the binomial distribution models the discrete number of successes (where we assume success is assigned a value of 1 and failure is assigned a value of 0), the Beta distribution is instead modelling the continuously defined probability of success, so probability is a random variable rather than a fixed parameter given or decided. In the Beta distribution, we take α to represent the fraction of the probability of success, and β to represent the fraction of the probability of failure (Kim, 2020).

This means that as a result, increasing α shifts the distribution to the right and therefore increases the expectation, whereas increasing β shifts to the left and decreases the expectation. When $\alpha = \beta$ the expectation will be 0.5 for all $\alpha, \beta \in \mathbb{R}$, and the standard deviation will decrease as α and β increase.

Chapter 4: Airy and Bessel Functions

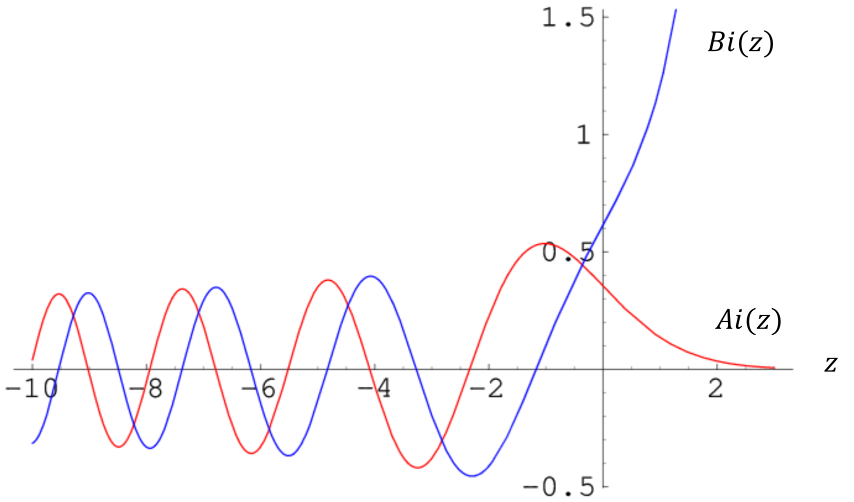
The Airy and Bessel functions are defined in terms of solutions to ODEs as the appropriate linearly independent solutions. They both have been extensively used in different areas of physics, which will be illustrated across the Chapter.

4.1 Introduction to Airy Functions

The two most common *Airy* functions are denoted Ai and Bi, which are entire functions (holomorphic over the complex plane) defined to be the two linearly independent solutions to the ODE below, which is referred to as the Airy differential equation, for $z \in \mathbb{C}$:

$$\frac{d^2y}{dz^2} - yz = 0.$$

The functions Ai and Bi are plotted below for $z \in \mathbb{R}$:



(Weisstein, 2022)

The Airy differential equation will have two linearly independent solutions which are oscillatory on the left half-plane and exponential on the right (growth for $Bi(z)$, decay for $Ai(z)$), as can be seen in the graph above. At $z = 0$, the constant values in the solution of the ODE have been chosen by asymptotic matching in order to ensure the solution maintains continuity and differentiability (Cook, 2013).

The function $Ai(z)$ in particular can be expressed as a closed form integral as seen below, for $z \in \mathbb{C}$:

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\frac{i(zt+t^3)}{3}}\, dt.$$

(Weisstein, 2022)

The Airy functions can also be represented in terms of the Gamma function at the point $z = 0$:

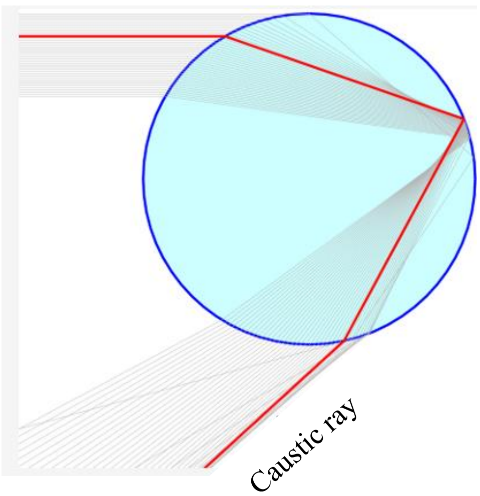
$$Ai(0) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)},$$

$$Bi(0) = \frac{1}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)}.$$

These values prove useful when solving the Airy ODE with initial conditions imposed at the origin.

4.2 Airy Functions and Supernumerary Fringes

Astronomer George Biddell Airy created the Airy function in an attempt to distinguish the different colours in a rainbow by looking at the transitions, or *supernumerary fringes*, between them (Freiberger, 2021). By finding the exact location of these supernumerary fringes, the information could be applied to develop telescope optics, by improving the resolution and ability to focus on different light intensities without needing to heavily readjust the equipment each time. The basic theory builds on the idea that the raindrops that act as the building blocks for the rainbow refract the caustic ray (as indicated in red) at different angles, which is dependent on the unique wavelength of each colour.



(Casselman, 2010)

In this situation the Airy function $Ai(z)$, which solves the ODE $y'' - yz = 0$, is used in a modified integral form, for $z \in \mathbb{R}$:

$$y(z) = Ai(z) = \int_0^\infty \cos\left(\frac{\pi}{2}\left(\frac{t^3}{3} + zt\right)\right) dt.$$

(Freiberger, 2021)

This is known as Airy's integral formula (Casselman, 2010) and when solved provides a value for the amplitude of a light wave found near the rainbow's caustic ray.

However, in practice, Airy encountered complications as a solution could only be found to two zeroes before the calculations became inaccurate (Hoffer, 2020). In 1850, George Stokes adapted the method through changing variables, rescaling and manipulating to achieve a solution with a much higher level of precision for the purposes of numerical computation and approximation, and therefore had the ability to confidently determine more digits in the values needed.

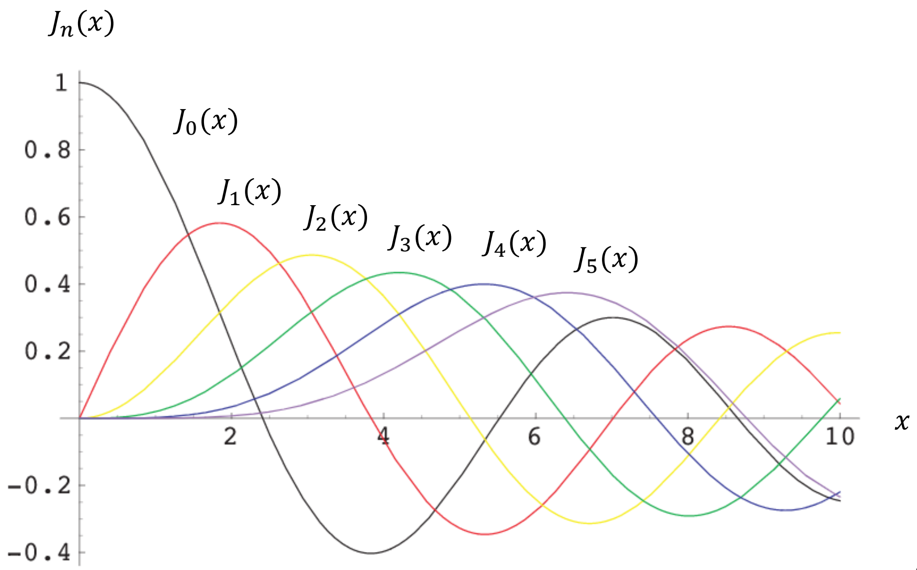
4.3 Introduction to Bessel Functions

For $n \in \mathbb{N}$, the n^{th} order Bessel differential equation is given by, for $x \geq 0$:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0.$$

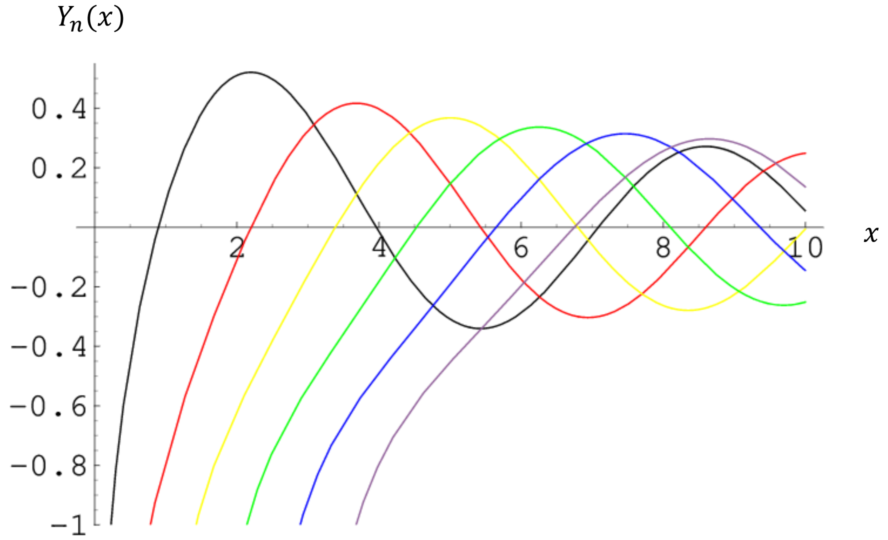
(Lambers, 2013)

As this differential equation is second order we know it must have two linearly independent solutions, in this case, these solutions are known as the *Bessel* functions (Niedziela, 2008). The Bessel function of the first kind is a solution which is non-singular at $x = 0$ and is denoted $J_n(x)$. Below is a plot for $J_n(x)$ for $x > 0$ and $n \in \{0, \dots, 5\}$:



(Weisstein, 2022)

The Bessel function of the second kind forms the second solution which is singular at $x = 0$, and is denoted $Y_n(x)$ (Lambers, 2013). They are also known as Neumann functions (Niedziela, 2008). A similar plot is shown below, where the colours are consistent with those from the previous plot:



(Weisstein, 2022)

The Bessel functions of the first and second kind can be combined in a certain form to yield Bessel functions of the third kind, otherwise known as Hankel functions, for $n \in \mathbb{N}$ (Harris, 2014):

$$H_n^{(1)}(x) = J_n(x) + iY_n(x)$$

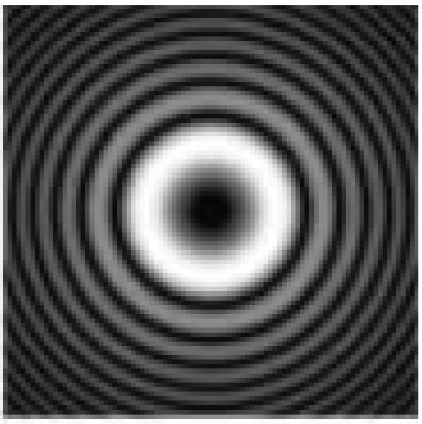
$$H_n^{(2)}(x) = J_n(x) - iY_n(x),$$

where $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are complex conjugates of one another, provided $x \geq 0$. Hankel functions are used to explain the curves of the cylindrical wave equation and therefore are required to express solutions, and when investigating whether any wavefronts appear to decay, or grow due to an increasing circumference, over a certain distance (Williams, 1999).

4.4 Electron Bessel Beams

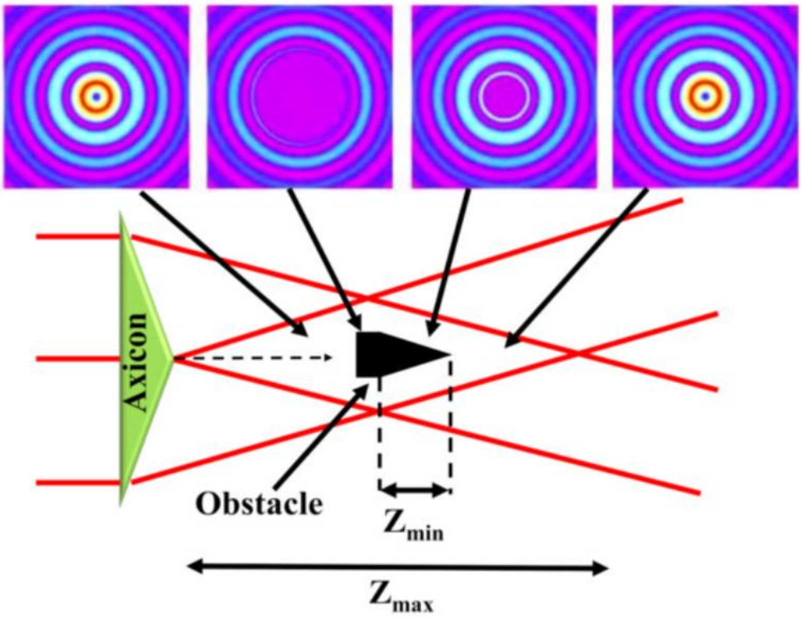
Bessel beams are known in optical physics as the light field that does not diffract (Dholakia, McGloin, 2004). Physicists have taken advantage of this property as beams that do not diffract can be used as instruments that can confine atomic particles by reduced channelling and sort particles into different classifications (Grillo, 2015). This property also means that after being subjected to an obstacle, Bessel beams are able to recover and revert back to their original form after some distance.

As the name would suggest, the Bessel beam is defined using the Bessel function, as the Bessel function of the first kind represents the amplitude of the beam. Bessel beams are thought to have a cross-sectional profile consisting of concentric rings (Dholakia, McGloin, 2004). The intensity profile of a J_1 beam is shown below:



(Dholakia, McGloin, 2004)

The intensity profile before, during, and after the passing of an obstacle can also be shown:



(Ali, 2020)

It can be seen above that the first and last intensity profile are the same, and the beam is unaffected after passing the obstacle. The beam starts to recover as soon as the obstacle narrows.

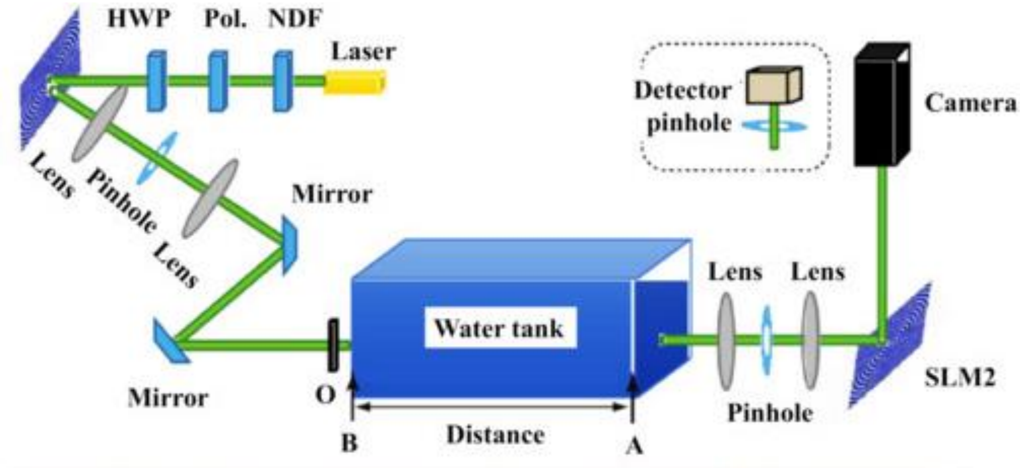
A particularly useful application of the Bessel beam is in electron microscopy, as the immunity to diffraction (which holds to some degree) results in a higher resolution and better quality imaging, meaning smaller changes in electron movement can be detected (Grillo, 2015).

Consequently, Bessel beams are beneficial in environments such as hospitals, where microscopy on the body is carried out on a small scale in a living cellular host (Gao, 2014). The clearer image provided enables doctors and other medical professionals to make judgements or give a diagnosis more confidently.

There are also (fairly recent) suggestions to start using Bessel beams for 3D printing of organs to be implanted into humans (O'Neal, 2020), and thus accelerate the development of bioprinting. Alongside the advantage of having a better resolution, Bessel beams were also found to result in a significantly reduced printing time, cutting the fabrication time by over half.

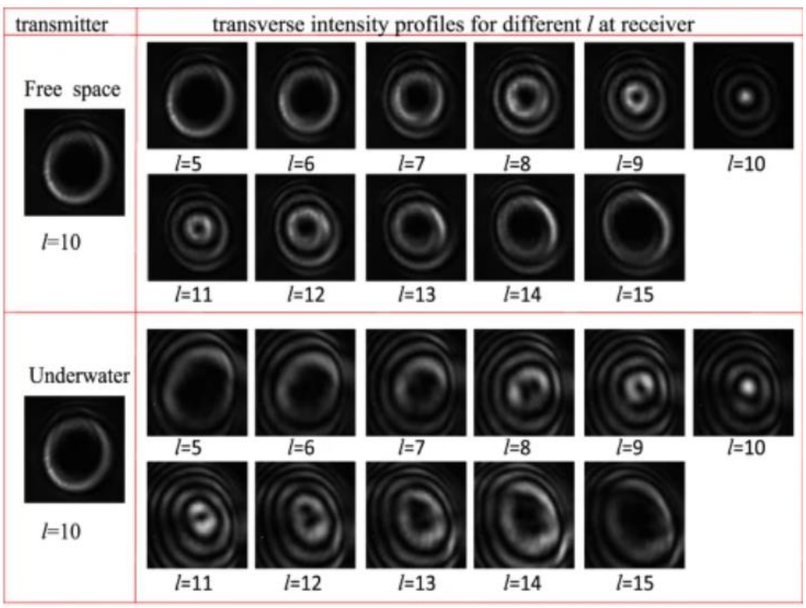
Furthermore, the ability for the beam to self-recover in the presence of obstacles enables the beams to overcome the varied obstacles and temperatures under the sea, and as such is used for under water communication (Ali, 2020). In this environment there are substantial amounts of data that need to be transmitted over large distances, and an increasing demand for this more reliable method due to an increase in divers, underwater sensors, submarines, and other manned and unmanned vehicles (Zhao, 2019).

The quality of Bessel beams for these situations can be experimented as shown in the set up below:



(Zhao, 2019)

Intensity profiles can be recorded from beams that are passed through the water tank and compared with undisturbed equivalent profiles to see the extent of corruption, and measure the probability of the beam being correctly detected and translated. The results from this particular experiment are as follows:



(Zhao, 2019)

It can be seen that although there is a noticeable difference when compared to the intensity profiles from free space, there are no complete blockages and for each integer value of l, which is defined as the topological charge, the profiles are sufficiently unique to not cause confusion. However, the results are of course dependent on the conditions, including factors such as the temperature, salinity of the water, presence of bubbles, turbulence, pollution, dissolved substances, currents, and riptides (Wang, 2020).

4.5 Relationship Between Airy and Bessel Functions

The Airy and Bessel functions may seem distinct but can be linked together, to see this we first need to define the modified Bessel function.

For $\nu \in \mathbb{C}$, define the *modified* Bessel function of the first kind (of order ν), $I_\nu : \mathbb{R} \to \mathbb{R}$, as:

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+\upsilon}}{n!\,\Gamma(\nu + n + 1)}.$$

(Baricz, 2010)

Using this as well as the definition of a Bessel function of the first kind, defined earlier as $J_n(x)$, the Airy function can be written in terms of $J_n(x)$ and $I_\nu(x)$ as follows:

$$Ai(x) = \begin{cases} \frac{\sqrt{x}}{3}\left(I_{-\frac{1}{3}}(\hat{x}) - I_{\frac{1}{3}}(\hat{x})\right) & if\ x \geq 0 \\ \frac{\sqrt{-x}}{3}\left(J_{-\frac{1}{3}}(\hat{x}) + J_{\frac{1}{3}}(\hat{x})\right) & if\ x < 0, \end{cases}$$

$$\text{with } \hat{x} = \tfrac{2}{3}\left(\sqrt{|x|}\right)^3.$$

(Cook, 2013)

Chapter 5: Möbius and Liouville Functions

In this final Chapter we will investigate two functions that are underpinned by number theory as they both use the notion of primes in their definitions. Firstly, the Möbius function will be covered, and will then be explored further using the application of graph theory to give a more visual demonstration. Secondly, the Liouville function will be introduced in order to show the importance of providing a formal proof for conjectures before believing them to be true.

5.1 Möbius Function

The Möbius function evaluates real integers in relation to their prime decomposition, with the range being the set $\{-1, 0, 1\}$, and is defined below for $n \in \mathbb{N}$:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^t & \text{if } n = p_1 p_2 \dots p_t \text{ with } p_i \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

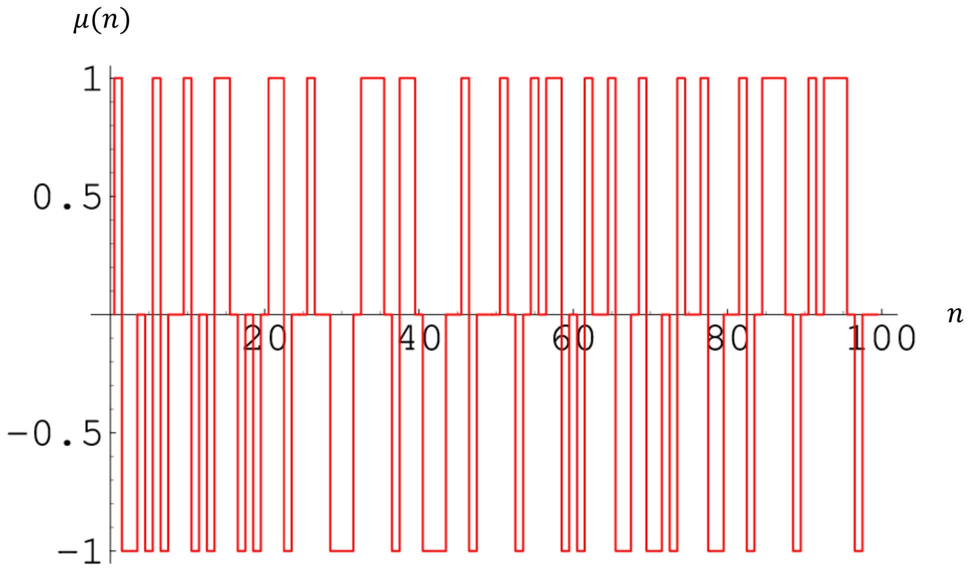
(Raji, 2021)

Therefore, for all primes p:

$$\mu(p) = -1,$$

whereas $\mu(n) = 0$ means that n must contain a square as one of its factors (Havil, 2003, pp.208-209).

The plot of the Möbius function for $n \in \{1, \dots, 100\}$ is shown below, with the points connected for clarity:



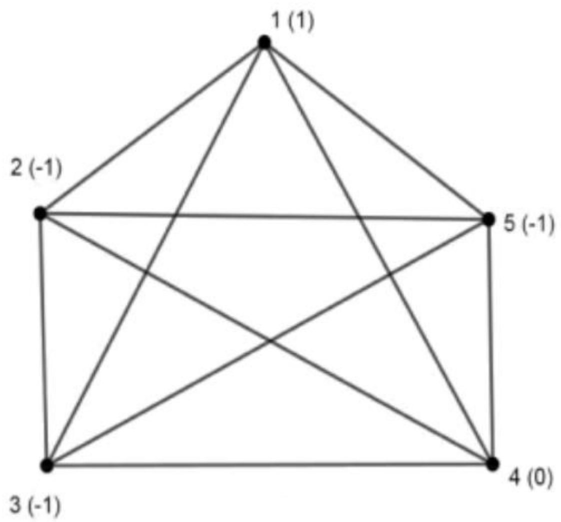
(Weisstein, 2022)

The Möbius function can be applied to graph theory by using a vertex set of $V = \{1, \dots, n\}$, and the edge set defined as:

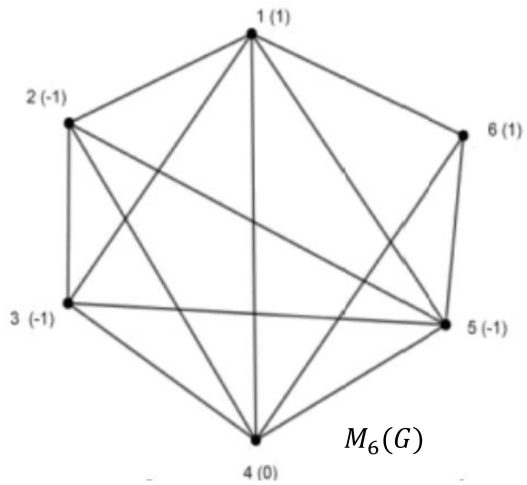
$$E = \{uv \mid \mu(uv) = \mu(u)\mu(v) \text{ where } u, v \in V\}.$$

(Aravinth, Vignesh, 2019)

This definition shows where the function satisfies the multiplicity property and generates the following graphs for $n = 5,6$:



$M_5(G)$



$M_6(G)$

(Aravinth, Vignesh, 2019)

When $n = 5,6$ the only edges that aren't included in the graphs is in the case for $n = 6$, where the missing edges are the ones required to connect vertices 2,6 and 3,6. This is because:

$$\mu(2 \times 6) = \mu(12) = 0 \neq -1 = -1 \times 1 = \mu(2)\mu(6).$$

Similarly:

$$\mu(3 \times 6) = \mu(18) = 0 \neq -1 = -1 \times 1 = \mu(3)\mu(6).$$

This is because the multiplication on the left hand side produces a number which has a square factor:

$$12 = 2^2 \times 3, \; 18 = 3^2 \times 2.$$

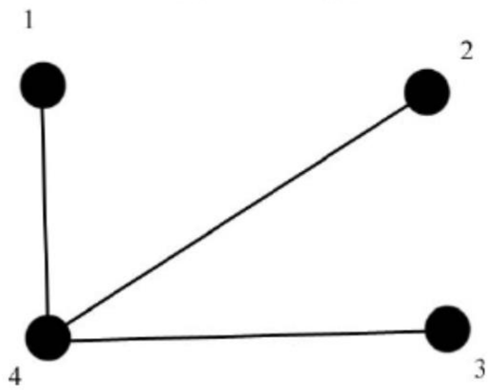
Therefore, from the definition of the function, the value is zero, and thus the multiplicative property does not hold.

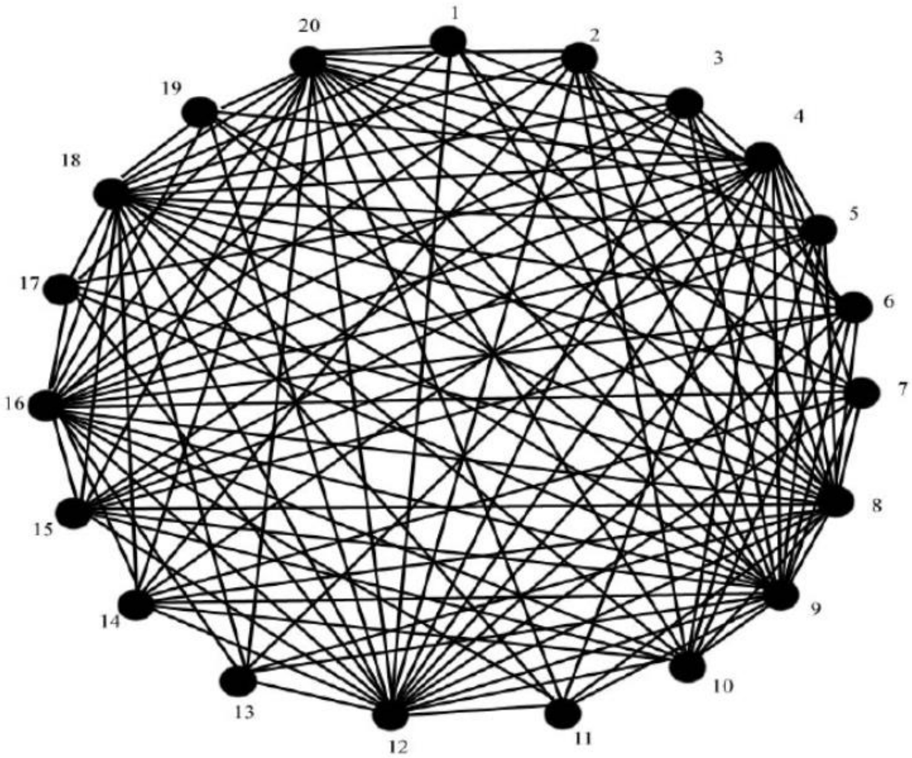
An alternative definition of the edge set is:

$$E = \{uv \mid \mu(uv) = 0 \text{ where } u, v \in V\}.$$

(Srimitra, 2017)

Examples for $n = 4$ and $n = 20$ are shown:





(Srimitra, 2017)

It is evident that square numbers, or numbers that have a square number as a factor, will be connected to all other vertices in the graph other than themselves (there are no self-loops) and there is no connection between any two vertices when neither has square factors. Therefore, in the case of $n \leq 3$, the graph with this edge set will be disconnected.

5.2 Liouville Function

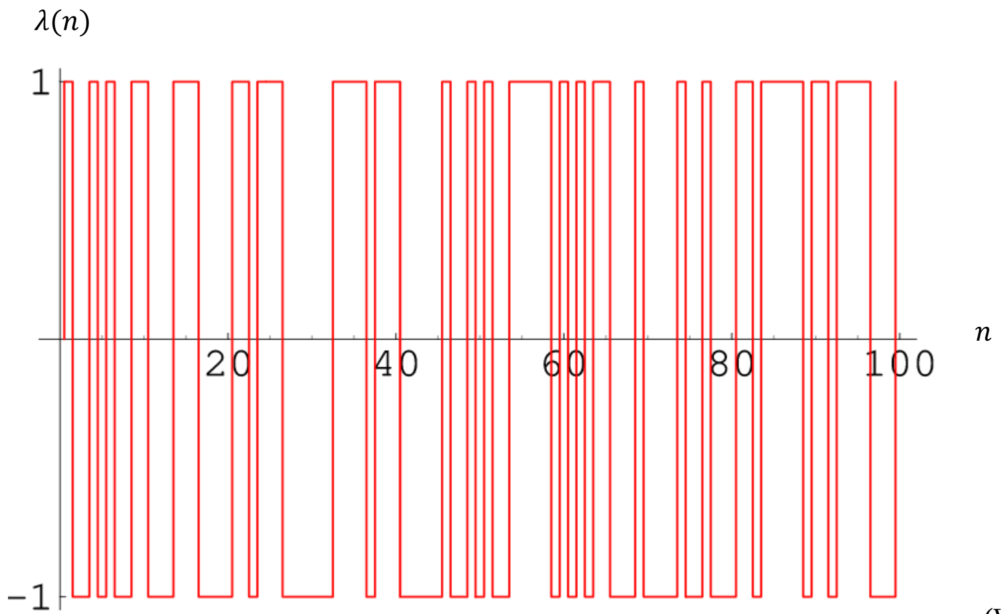
The Liouville function is defined, for $n \in \mathbb{N}$, as:

$$\lambda(n) = (-1)^{\Omega(n)},$$

(Weisstein, 2022)

where $\Omega(n)$ is the number of prime factors of n (not necessarily distinct).

The plot for $n \in \{1, \dots, 100\}$ is shown below:



(Weisstein, 2022)

The Liouville function was used in the Pólya conjecture, where it was stated that the function $L: \mathbb{N} \to \mathbb{Z}$ given by:

$$L(x) = \sum_{n=2}^{x} \lambda(n)$$

is negative for all $x > 1$ (Lehman, 1960).

This statement is conveying the idea that for any integer x, greater than 1, at least half of any numbers in the range of $[2, x]$ will have an odd number of primes in its unique prime decomposition (Krivyakov, 2019).

For example, for $n = 8$:

	2	3	4	5	6	7	8
Prime decomposition	2	3	2×2	5	2×3	7	2×2×2
Number of primes	1	1	2	1	2	1	3
Odd or even?	Odd	Odd	Even	Odd	Even	Odd	Odd

As a result, we can see that for $n = 8$, five integers have an odd number of primes in their decomposition and two integers have an even number of primes, so consequently the conjecture holds true in this case.

In fact, the conjecture is true up until $n = 906,150,257$ which is the first counter-example (Darling, 2016), and was found in 1958, almost 40 years after the conjecture was first posited. Many other counter-examples greater than 906,150,257 can also be found.

The disproven conjecture is now used as an example of how a hypothesised statement can be seemingly true and misleadingly hold for all 'small' numbers but still in fact turn out to be false (Stein, 1999, p.483), and as a result, it is not reasonable to assume that a relationship will hold true for all integers n without proof as the values of n grow indefinitely, and there is no upper bound on where the location of the smallest counter-example will be. Through the Pólya conjecture, the Liouville function has contributed to this argument.

5.3 Relationships to the Riemann Zeta Function

Both the Möbius and Liouville functions can be linked to the Riemann zeta function (which was covered in Chapter 2) by the following relations, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Firstly, for the Möbius function:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

(Corn, 2022)

Secondly, for the Liouville function:

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

(Lehman, 1960)

This provides another example of special functions being interconnected, even across the Chapters of this project.

Chapter 6: Conclusion

There are examples of special functions that arise in a number of different ways, firstly from the form of generalised distributions which can be expressed with simple definitions, and the Airy and Bessel functions that are the linearly independent solutions of certain ODEs. We have also seen that the commonly used factorial function can be extended to give the Gamma function, and using the concept of prime numbers and unique prime decompositions of integers can lead to the Möbius and Liouville functions.

These special functions have been shown to have a wide range of applications covering many areas of mathematics and even extending to related subjects, including statistics, physics, and medicine. Many of the applications used in developing technology or aiding in experiments are still ongoing today, meaning the functions will continue to be beneficial in many fields in the years ahead. The demonstration of how versatile the functions are shows explicitly how extensions of mathematical concepts into a defined mapping does have links to other topics, sometimes even in cases where there was no initial intention of the theory being applied to more tangible concepts, and sometimes the link between the functions and applications has been seemingly coincidental.

In addition to the special functions having many relationships to other areas of study and therefore applications, there are also many ways to express two (or more, with increasing complexity) of these functions in the same equation, even in cases where the specific functions used were defined from different notions and may seem from disjoint areas of mathematics, proving further how unrestricted special functions are.

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Appendix A

$$f(n) = \int_0^1 (-\ln(x))^n \, dx.$$

Using the substitution:

$$x = e^{-t} \Rightarrow t = -\ln(x),$$

$$dx = -e^{-t}\, dt,$$

This gives:

$$f(n) = \int_\infty^0 -t^n e^{-t}\, dt = \int_0^\infty t^n e^{-t}\, dt.$$

Using integration by parts:

$$u = t^n, \; du = nt^{n-1}\, dt,$$

$$v = -e^{-t}, \; dv = e^{-t}\, dt$$

$$= [-t^n e^{-t}]_0^\infty + n\int_0^\infty t^{n-1} e^{-t}\, dt = nf(n-1)$$

Therefore:

$$f(n) = nf(n-1) = n(n-1)f(n-2) = \cdots = n(n-1)(n-2)(\ldots)(2)(1)f(0) = n!,$$

With the final equality following from:

$$f(0) = 1.$$

Therefore:

$$n! = \int_0^1 (-\ln(x))^n\, dx.$$

Appendix B

Using integration by parts:

$$\Pi(n) = \int_0^\infty t^n e^{-t}\, dt$$

$$= [-t^n e^{-t}]_0^\infty - \int_0^\infty -nt^{n-1} e^{-t}\, dt$$

$$= n \int_0^\infty t^{n-1} e^{-t}\, dt$$

$$= n\, \Pi(n-1).$$

Then, using proof by induction:

Base case:

$$\Pi(1) = \int_0^\infty t e^{-t}\, dt$$

$$= [-t e^{-t}]_0^\infty + \int_0^\infty e^{-t}\, dt$$

$$= 1.$$

Therefore:

$$\Pi(1) = 1 = 1!.$$

Now, to extend, assume that:

$$\Pi(n-1) = (n-1)!,$$

Then:

$$\Pi(n) = n\Pi(n-1) = n(n-1)! = n!,$$

Thus the proof is completed (Müller, 2021).